

Nonlinear Control

Lecture # 10

Time Varying
and

Perturbed Systems

Boundedness and Ultimate Boundedness

Definition 4.3

The solutions of $\dot{x} = f(t, x)$ are

- uniformly bounded if there exists $c > 0$, independent of t_0 , and for every $a \in (0, c)$, there is $\beta > 0$, dependent on a but independent of t_0 , such that

$$\|x(t_0)\| \leq a \Rightarrow \|x(t)\| \leq \beta, \quad \forall t \geq t_0$$

- uniformly ultimately bounded with ultimate bound b if there exists a positive constant c , independent of t_0 , and for every $a \in (0, c)$, there is $T \geq 0$, dependent on a and b but independent of t_0 , such that

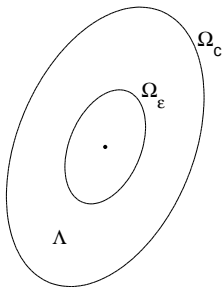
$$\|x(t_0)\| \leq a \Rightarrow \|x(t)\| \leq b, \quad \forall t \geq t_0 + T$$

- Add “Globally” if a can be arbitrarily large
- Drop “uniformly” if $\dot{x} = f(x)$

Lyapunov Analysis: Let $V(x)$ be a cont. diff. positive definite function and suppose the sets

$$\Omega_c = \{V(x) \leq c\}, \quad \Omega_\varepsilon = \{V(x) \leq \varepsilon\}, \quad \Lambda = \{\varepsilon \leq V(x) \leq c\}$$

are compact for some $c > \varepsilon > 0$



Suppose

$$\dot{V}(t, x) = \frac{\partial V}{\partial x} f(t, x) \leq -W_3(x), \quad \forall x \in \Lambda, \quad \forall t \geq 0$$

$W_3(x)$ is continuous and positive definite

Ω_c and Ω_ε are positively invariant

$$k = \min_{x \in \Lambda} W_3(x) > 0$$

$$\dot{V}(t, x) \leq -k, \quad \forall x \in \Lambda, \quad \forall t \geq t_0 \geq 0$$

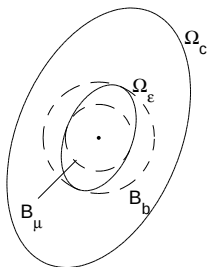
$$V(x(t)) \leq V(x(t_0)) - k(t - t_0) \leq c - k(t - t_0)$$

$x(t)$ enters the set Ω_ε within the interval $[t_0, t_0 + (c - \varepsilon)/k]$

Suppose

$$\dot{V}(t, x) \leq -W_3(x), \quad \forall x \in D \text{ with } \|x\| \geq \mu, \quad \forall t \geq 0$$

Choose c and ε such that $\Lambda \subset D \cap \{\|x\| \geq \mu\}$



Let α_1 and α_2 be class $\mathcal{K}$ functions such that

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$$

$$V(x) \leq c \Rightarrow \alpha_1(\|x\|) \leq c \Leftrightarrow \|x\| \leq \alpha_1^{-1}(c)$$

$$\text{If } B_r \subset D, \quad c = \alpha_1(r) \Rightarrow \Omega_c \subset B_r \subset D$$

$$\|x\| \leq \mu \Rightarrow V(x) \leq \alpha_2(\mu)$$

$$\varepsilon = \alpha_2(\mu) \Rightarrow B_\mu \subset \Omega_\varepsilon$$

What is the ultimate bound?

$$V(x) \leq \varepsilon \Rightarrow \alpha_1(\|x\|) \leq \varepsilon \Leftrightarrow \|x\| \leq \alpha_1^{-1}(\varepsilon) = \alpha_1^{-1}(\alpha_2(\mu))$$

Theorem 4.4

Suppose $B_\mu \subset D \subset \mathbb{R}^n$ and

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$$

$$\frac{\partial V}{\partial x} f(t, x) \leq -W_3(x), \quad \forall x \in D \text{ with } \|x\| \geq \mu, \quad \forall t \geq 0$$

where α_1 and α_2 are class $\mathcal{K}$ functions and $W_3(x)$ is a continuous positive definite function. Choose $c > 0$ such that $\Omega_c = \{V(x) \leq c\}$ is compact and contained in D and suppose $\mu < \alpha_2^{-1}(c)$. Then, Ω_c is positively invariant and there exists a class $\mathcal{KL}$ function β such that for every $x(t_0) \in \Omega_c$,

$$\|x(t)\| \leq \max \left\{ \beta(\|x(t_0)\|, t - t_0), \alpha_1^{-1}(\alpha_2(\mu)) \right\}, \quad \forall t \geq t_0$$

If $D = \mathbb{R}^n$ and $\alpha_1 \in \mathcal{K}_\infty$, the inequality holds $\forall x(t_0), \forall \mu$

Remarks

- The ultimate bound is independent of the initial state
- The ultimate bound is a class $\mathcal{K}$ function of μ ; hence, the smaller the value of μ , the smaller the ultimate bound. As $\mu \rightarrow 0$, the ultimate bound approaches zero

Example 4.8

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -(1 + x_1^2)x_1 - x_2 + M \cos \omega t, \quad M \geq 0$$

With $M = 0$, $\dot{x}_2 = -(1 + x_1^2)x_1 - x_2 = -h(x_1) - x_2$

$$V(x) = x^T \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} x + 2 \int_0^{x_1} (y + y^3) dy \quad (\text{Example 3.7})$$

$$V(x) = x^T \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} x + \frac{1}{2}x_1^4 \stackrel{\text{def}}{=} x^T P x + \frac{1}{2}x_1^4$$

$$\lambda_{\min}(P)\|x\|^2 \leq V(x) \leq \lambda_{\max}(P)\|x\|^2 + \frac{1}{2}\|x\|^4$$

$$\alpha_1(r) = \lambda_{\min}(P)r^2, \quad \alpha_2(r) = \lambda_{\max}(P)r^2 + \frac{1}{2}r^4$$

$$\begin{aligned} \dot{V} &= -x_1^2 - x_1^4 - x_2^2 + (x_1 + 2x_2)M \cos \omega t \\ &\leq -\|x\|^2 - x_1^4 + M\sqrt{5}\|x\| \\ &= -(1-\theta)\|x\|^2 - x_1^4 - \theta\|x\|^2 + M\sqrt{5}\|x\| \\ &\quad (0 < \theta < 1) \\ &\leq -(1-\theta)\|x\|^2 - x_1^4, \quad \forall \|x\| \geq M\sqrt{5}/\theta \stackrel{\text{def}}{=} \mu \end{aligned}$$

The solutions are GUUB by

$$b = \alpha_1^{-1}(\alpha_2(\mu)) = \sqrt{\frac{\lambda_{\max}(P)\mu^2 + \mu^4/2}{\lambda_{\min}(P)}}$$

Theorem 4.5

Suppose

$$c_1 \|x\|^2 \leq V(x) \leq c_2 \|x\|^2$$

$$\frac{\partial V}{\partial x} f(t, x) \leq -c_3 \|x\|^2, \quad \forall x \in D \text{ with } \|x\| \geq \mu, \quad \forall t \geq 0$$

for some positive constants c_1 to c_3 , and $\mu < \sqrt{c/c_2}$. Then, $\Omega_c = \{V(x) \leq c\}$ is positively invariant and $\forall x(t_0) \in \Omega_c$

$$V(x(t)) \leq \max \left\{ V(x(t_0)) e^{-(c_3/c_2)(t-t_0)}, c_2 \mu^2 \right\}, \quad \forall t \geq t_0$$

$$\|x(t)\| \leq \sqrt{c_2/c_1} \max \left\{ \|x(t_0)\| e^{-(c_3/c_2)(t-t_0)/2}, \mu \right\}, \quad \forall t \geq t_0$$

If $D = R^n$, the inequalities hold $\forall x(t_0), \forall \mu$

Example 4.9

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -h(x_1) - x_2 + u(t), \quad h(x_1) = x_1 - \frac{1}{3}x_1^3$$

$$|u(t)| \leq d$$

$$V(x) = \frac{1}{2}x^T \begin{bmatrix} k & k \\ k & 1 \end{bmatrix} x + \int_0^{x_1} h(y) dy, \quad 0 < k < 1$$

$$\frac{2}{3}x_1^2 \leq x_1 h(x_1) \leq x_1^2, \quad \frac{5}{12}x_1^2 \leq \int_0^{x_1} h(y) dy \leq \frac{1}{2}x_1^2, \quad \forall |x_1| \leq 1$$

$$\lambda_{\min}(P_1)\|x\|^2 \leq x^T P_1 x \leq V(x) \leq x^T P_2 x \leq \lambda_{\max}(P_2)\|x\|^2$$

$$P_1 = \frac{1}{2} \begin{bmatrix} k + \frac{5}{6} & k \\ k & 1 \end{bmatrix}, \quad P_2 = \frac{1}{2} \begin{bmatrix} k + 1 & k \\ k & 1 \end{bmatrix}$$

$$\begin{aligned}\dot{V} &= -kx_1h(x_1) - (1-k)x_2^2 + (kx_1 + x_2)u(t) \\ &\leq -\frac{2}{3}kx_1^2 - (1-k)x_2^2 + |kx_1 + x_2| d\end{aligned}$$

$$k = \frac{3}{5} \Rightarrow c_1 = \lambda_{\min}(P_1) = 0.2894, \quad c_2 = \lambda_{\max}(P_2) = 0.9854$$

$$\begin{aligned}\dot{V} &\leq -\frac{0.1 \times 2}{5} \|x\|^2 - \frac{0.9 \times 2}{5} \|x\|^2 + \sqrt{1 + \left(\frac{3}{5}\right)^2} \|x\| d \\ &\leq \frac{0.1 \times 2}{5} \|x\|^2, \quad \forall \|x\| \geq 3.2394 d \stackrel{\text{def}}{=} \mu\end{aligned}$$

$$c = \min_{|x_1|=1} V(x) = 0.5367 \Rightarrow \Omega_c = \{V(x) \leq c\} \subset \{|x_1| \leq 1\}$$

For $\mu < \sqrt{c/c_2}$ we need $d < 0.2278$. Theorem 4.5 holds and
 $b = \mu\sqrt{c_2/c_1} = 5.9775 d$

Perturbed Systems: Nonvanishing Perturbation

Nominal System:

$$\dot{x} = f(x), \quad f(0) = 0$$

Perturbed System:

$$\dot{x} = f(x) + g(t, x), \quad g(t, 0) \neq 0$$

Case 1:(Lemma 4.3) The origin of $\dot{x} = f(x)$ is exponentially stable

Case 2:(Lemma 4.4) The origin of $\dot{x} = f(x)$ is asymptotically stable

Lemma 4.3

Suppose that $\forall x \in B_r, \forall t \geq 0$

$$c_1\|x\|^2 \leq V(x) \leq c_2\|x\|^2$$

$$\frac{\partial V}{\partial x} f(x) \leq -c_3\|x\|^2, \quad \left\| \frac{\partial V}{\partial x} \right\| \leq c_4\|x\|$$

$$\|g(t, x)\| \leq \delta < \frac{c_3}{c_4} \sqrt{\frac{c_1}{c_2}} \theta r, \quad 0 < \theta < 1$$

Then, for all $x(t_0) \in \{V(x) \leq c_1 r^2\}$

$$\|x(t)\| \leq \max \{k \exp[-\gamma(t - t_0)]\|x(t_0)\|, b\}, \quad \forall t \geq t_0$$

$$k = \sqrt{\frac{c_2}{c_1}}, \quad \gamma = \frac{(1 - \theta)c_3}{2c_2}, \quad b = \frac{\delta c_4}{\theta c_3} \sqrt{\frac{c_2}{c_1}}$$

Proof

Apply Theorem 4.5

$$\begin{aligned}\dot{V}(t, x) &= \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(t, x) \\ &\leq -c_3 \|x\|^2 + \left\| \frac{\partial V}{\partial x} \right\| \|g(t, x)\| \\ &\leq -c_3 \|x\|^2 + c_4 \delta \|x\| \\ &= -(1 - \theta) c_3 \|x\|^2 - \theta c_3 \|x\|^2 + c_4 \delta \|x\| \\ &\leq -(1 - \theta) c_3 \|x\|^2, \quad \forall \|x\| \geq \delta c_4 / (\theta c_3) \stackrel{\text{def}}{=} \mu\end{aligned}$$

$$x(t_0) \in \Omega = \{V(x) \leq c_1 r^2\}$$

$$\mu < r \sqrt{\frac{c_1}{c_2}} \Leftrightarrow \delta < \frac{c_3}{c_4} \sqrt{\frac{c_1}{c_2}} \theta r, \quad b = \mu \sqrt{\frac{c_2}{c_1}} \Leftrightarrow b = \frac{\delta c_4}{\theta c_3} \sqrt{\frac{c_2}{c_1}}$$

Example 4.10

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -4x_1 - 2x_2 + \beta x_2^3 + d(t)$$

$$\beta \geq 0, \quad |d(t)| \leq \delta, \forall t \geq 0$$

$$V(x) = x^T P x = x^T \begin{bmatrix} \frac{3}{2} & \frac{1}{8} \\ \frac{1}{8} & \frac{5}{16} \end{bmatrix} x \quad (\text{Example 4.5})$$

$$\begin{aligned} \dot{V}(t, x) &= -\|x\|^2 + 2\beta x_2^2 \left(\frac{1}{8}x_1x_2 + \frac{5}{16}x_2^2 \right) \\ &\quad + 2d(t) \left(\frac{1}{8}x_1 + \frac{5}{16}x_2 \right) \\ &\leq -\|x\|^2 + \frac{\sqrt{29}}{8} \beta k_2^2 \|x\|^2 + \frac{\sqrt{29}\delta}{8} \|x\| \end{aligned}$$

$$k_2 = \max_{x^T P x \leq c} |x_2| = 1.8194\sqrt{c}$$

Suppose $\beta \leq 8(1 - \zeta)/(\sqrt{29}k_2^2)$ ($0 < \zeta < 1$)

$$\begin{aligned} \dot{V}(t, x) &\leq -\zeta \|x\|^2 + \frac{\sqrt{29}\delta}{8} \|x\| \\ &\leq -(1 - \theta)\zeta \|x\|^2, \quad \forall \|x\| \geq \frac{\sqrt{29}\delta}{8\zeta\theta} \stackrel{\text{def}}{=} \mu \end{aligned} \quad (0 < \theta < 1)$$

If $\mu^2 \lambda_{\max}(P) < c$, then all solutions of the perturbed system, starting in Ω_c , are uniformly ultimately bounded by

$$b = \frac{\sqrt{29}\delta}{8\zeta\theta} \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}$$

Lemma 4.4

Suppose that $\forall x \in B_r, \forall t \geq 0$

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \quad \frac{\partial V}{\partial x} f(x) \leq -\alpha_3(\|x\|)$$

$$\left\| \frac{\partial V}{\partial x}(x) \right\| \leq k, \quad \|g(t, x)\| \leq \delta < \frac{\theta \alpha_3(\alpha_2^{-1}(\alpha_1(r)))}{k}$$

$\alpha_i \in \mathcal{K}, 0 < \theta < 1$. Then, $\forall x(t_0) \in \{V(x) \leq \alpha_1(r)\}$

$$\|x(t)\| \leq \max \{ \beta(\|x(t_0)\|, t - t_0), \rho(\delta) \}, \quad \forall t \geq t_0, \beta \in \mathcal{KL}$$

$$\rho(\delta) = \alpha_1^{-1} \left(\alpha_2 \left(\alpha_3^{-1} \left(\frac{\delta k}{\theta} \right) \right) \right)$$

Proof: Apply Theorem 4.4

Compare

Case 1: $\delta < \frac{c_3}{c_4} \sqrt{\frac{c_1}{c_2}} \theta r$

Case 2: $\delta < \frac{\theta \alpha_3(\alpha_2^{-1}(\alpha_1(r)))}{k}$

Example 4.11

$$\dot{x} = -\frac{x}{1+x^2} \quad (\text{Globally asymptotically stable})$$

$$V(x) = x^4 \Rightarrow \frac{\partial V}{\partial x} \left[-\frac{x}{1+x^2} \right] = -\frac{4x^4}{1+x^2}$$

$$\alpha_1(|x|) = \alpha_2(|x|) = |x|^4; \quad \alpha_3(|x|) = \frac{4|x|^4}{1+|x|^2}; \quad k = 4r^3$$

$$\frac{\theta \alpha_3(\alpha_2^{-1}(\alpha_1(r)))}{k} = \frac{\theta \alpha_3(r)}{k} = \frac{r\theta}{1+r^2} < \frac{1}{2}$$

$$\dot{x} = -\frac{x}{1+x^2} + \delta, \quad \delta > \frac{1}{2} \Rightarrow \lim_{t \rightarrow \infty} x(t) = \infty$$