Nonlinear Control

Lecture # 11

Time Varying

and

Perturbed Systems

Input-to-State Stability (ISS)

Definition 4.4

The system $\dot{x} = f(x, u)$ is input-to-state stable if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for any initial state $x(t_0)$ and any bounded input $u(t)$

$$\|x(t)\| \leq \max\left\{\beta(\|x(t_0)\|, t - t_0),\ \gamma\left(\sup_{t_0 \leq \tau \leq t} \|u(\tau)\|\right)\right\}$$

for all $t \geq t_0$

ISS of $\dot{x} = f(x, u)$ implies

- BIBS stability
- $x(t)$ is ultimately bounded by $\gamma\left(\sup_{t_0 \leq \tau \leq t} \|u(\tau)\|\right)$
- $\lim_{t\to\infty} u(t) = 0 \quad \Rightarrow \lim_{t\to\infty} x(t) = 0$
- The origin of $\dot{x} = f(x, 0)$ is GAS

Theorem 4.6

Let $V(x)$ be a continuously differentiable function

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$$

$$\frac{\partial V}{\partial x} f(x,u) \leq -W_3(x), \quad \forall \ \|x\| \geq \rho(\|u\|) > 0$$

$\forall \ x \in R^n, \ u \in R^m$, where $\alpha_1, \ \alpha_2 \in \mathcal{K}_\infty$, $\rho \in \mathcal{K}$, and $W_3(x)$ is a continuous positive definite function. Then, the system $\dot{x} = f(x,u)$ is ISS with $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$

Proof

Let $\mu = \rho(sup_{\tau \geq t_0} \|u(\tau)\|)$; then

$$\frac{\partial V}{\partial x} f(x,u) \leq -W_3(x), \quad \forall \ \|x\| \geq \mu$$

Apply Theorem 4.4

$$\|x(t)\| \le \max\left\{\beta(\|x(t_0)\|, t - t_0), \alpha_1^{-1}(\alpha_2(\mu))\right\}$$

$$\|x(t)\| \le \max\left\{\beta(\|x(t_0)\|, t - t_0), \gamma\left(\sup_{\tau \ge t_0} \|u(\tau)\|\right)\right\}$$

Since $x(t)$ depends only on $u(\tau)$ for $t_0 \le \tau \le t$, the supremum on the right-hand side can be taken over $[t_0, t]$

Lemma 4.5

Suppose $f(x, u)$ is continuously differentiable and globally Lipschitz in (x, u). If $\dot{x} = f(x, 0)$ has a globally exponentially stable equilibrium point at the origin, then the system $\dot{x} = f(x, u)$ is input-to-state stable

Proof: Apply (the converse Lyapunov) Theorem 3.8

Example 4.12

$$\dot{x} = -x^3 + u$$

The origin of $\dot{x} = -x^3$ is globally asymptotically stable

$$V = \tfrac{1}{2}x^2$$

$$\begin{aligned} \dot{V} &= -x^4 + xu \\ &= -(1-\theta)x^4 - \theta x^4 + xu \\ &\leq -(1-\theta)x^4, \quad \forall\ |x| \geq \left(\tfrac{|u|}{\theta}\right)^{1/3} \end{aligned}$$

$$0 < \theta < 1$$

The system is ISS with $\gamma(r) = (r/\theta)^{1/3}$

Example 4.13

$$\dot{x} = -x - 2x^3 + (1 + x^2)u^2$$

The origin of $\dot{x} = -x - 2x^3$ is globally exponentially stable

$$V = \frac{1}{2}x^2$$

$$\begin{aligned} \dot{V} &= -x^2 - 2x^4 + x(1 + x^2)u^2 \\ &= -x^4 - x^2(1 + x^2) + x(1 + x^2)u^2 \\ &\leq -x^4, \ \ \forall \ |x| \geq u^2 \end{aligned}$$

The system is ISS with $\gamma(r) = r^2$

Example 4.14

$$\dot{x}_1 = -x_1 + x_2, \qquad \dot{x}_2 = -x_1^3 - x_2 + u$$

Investigate GAS of $\quad \dot{x}_1 = -x_1 + x_2, \qquad \dot{x}_2 = -x_1^3 - x_2$

$$V(x) = \tfrac{1}{4}x_1^4 + \tfrac{1}{2}x_2^2 \quad \Rightarrow \quad \dot{V} = -x_1^4 - x_2^2$$

Now $u \neq 0$

$$\dot{V} = -x_1^4 - x_2^2 + x_2 u \leq -x_1^4 - x_2^2 + |x_2|\,|u|$$

$$\dot{V} \leq -(1-\theta)[x_1^4 + x_2^2] - \theta x_1^4 - \theta x_2^2 + |x_2|\,|u|$$

$$(0 < \theta < 1)$$

$-\theta x_2^2 + |x_2|\ |u| \leq 0$ for $|x_2| \geq |u|/\theta$ and has a maximum value of $u^2/(4\theta)$ for $|x_2| < |u|/\theta$

$$x_1^2 \geq \frac{|u|}{2\theta} \ \text{ or } \ x_2^2 \geq \frac{u^2}{\theta^2} \ \Rightarrow \ -\theta x_1^4 - \theta x_2^2 + |x_2|\ |u| \leq 0$$

$$\|x\|^2 \geq \frac{|u|}{2\theta} + \frac{u^2}{\theta^2} \ \Rightarrow \ -\theta x_1^4 - \theta x_2^2 + |x_2|\ |u| \leq 0$$

$$\rho(r) = \sqrt{\frac{r}{2\theta} + \frac{r^2}{\theta^2}}$$

$$\dot{V} \leq -(1-\theta)[x_1^4 + x_2^2], \ \ \forall\ \|x\| \geq \rho(|u|)$$

The system is ISS

Lemma 4.6

If the systems $\dot{\eta} = f_1(\eta, \xi)$ and $\dot{\xi} = f_2(\xi, u)$ are input-to-state stable, then the cascade connection

$$\dot{\eta} = f_1(\eta, \xi), \qquad \dot{\xi} = f_2(\xi, u)$$

is input-to-state stable. Consequently, If $\dot{\eta} = f_1(\eta, \xi)$ is input-to-state stable and the origin of $\dot{\xi} = f_2(\xi)$ is globally asymptotically stable, then the origin of the cascade connection

$$\dot{\eta} = f_1(\eta, \xi), \qquad \dot{\xi} = f_2(\xi)$$

is globally asymptotically stable

Example 4.15

$$\dot{x}_1 = -x_1 + x_2^2, \qquad \dot{x}_2 = -x_2 + u$$

The system $\dot{x}_1 = -x_1 + x_2^2$ is input-to-state stable, as seen from Theorem 4.6 with $V(x_1) = \frac{1}{2}x_1^2$

$$\dot{V} = -x_1^2 + x_1 x_2^2 \le -(1-\theta)x_1^2, \text{ for } |x_1| \ge x_2^2/\theta, \ 0 < \theta < 1$$

The linear system $\dot{x}_2 = -x_2 + u$ is input-to-state stable by Lemma 4.5

The cascade connection is input-to-state stable

Definition 4.5

Let $\mathcal{X} \subset R^n$ and $\mathcal{U} \subset R^m$ be bounded sets containing their respective origins as interior points. The system $\dot{x} = f(x, u)$ is regionally input-to-state stable with respect to $\mathcal{X} \times \mathcal{U}$ if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for any initial state $x(t_0) \in \mathcal{X}$ and any input u with $u(t) \in \mathcal{U}$ for all $t \geq t_0$, the solution $x(t)$ belongs to $\mathcal{X}$ for all $t \geq t_0$ and satisfies

$$\|x(t)\| \leq \max\left\{\beta(\|x(t_0)\|, t - t_0),\ \gamma\left(\sup_{t_0 \leq \tau \leq t} \|u(\tau)\|\right)\right\}$$

The system $\dot{x} = f(x, u)$ is locally input-to-state stable if it is regionally input-to-state stable with respect to some neighborhood of the origin $(x = 0, u = 0)$

Theorem 4.7

Suppose $f(x,u)$ is locally Lipschitz in (x,u) for all $x \in B_r$ and $u \in B_\lambda$. Let $V(x)$ be a continuously differentiable function that satisfies

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$$

$$\frac{\partial V}{\partial x} f(x,u) \leq -W_3(x), \quad \forall \; \|x\| \geq \rho(\|u\|) > 0$$

for all $x \in B_r$ and $u \in B_\lambda$, where α_1, α_2, $\rho \in \mathcal{K}$ and $W_3(x)$ is a continuous positive definite function. Suppose $\alpha_1(r) > \alpha_2(\rho(\lambda))$ and let $\Omega = \{V(x) \leq \alpha_1(r)\}$. Then, the system $\dot{x} = f(x,u)$ is regionally input-to-state stable with respect to $\Omega \times B_\lambda$ and $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$

Local input-to-state stability of $\dot{x} = f(x, u)$ is equivalent to asymptotic stability of the origin of $\dot{x} = f(x, 0)$

Lemma 4.7

Suppose $f(x, u)$ is locally Lipschitz in (x, u) in some neighborhood of $(x = 0, u = 0)$. Then, the system $\dot{x} = f(x, u)$ is locally input-to-state stable if and only if the unforced system $\dot{x} = f(x, 0)$ has an asymptotically stable equilibrium point at the origin

The proof uses (converse Lyapunov) Theorem 3.9