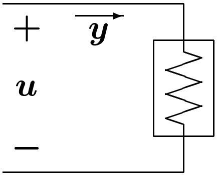
Nonlinear Control

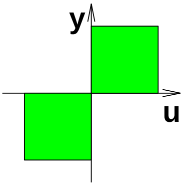
Lecture # 12

Passivity

Memoryless Functions



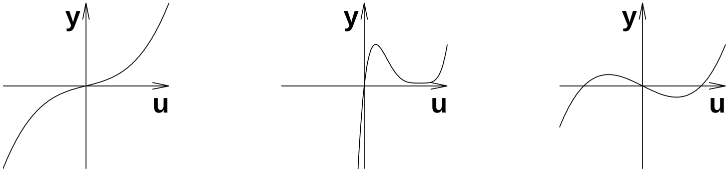
(a)



(b)

$$\text{power inflow} \; = \; uy$$

Resistor is passive if $uy \geq 0$



(a) (b) (c)

Passive Passive Not passive

$$y = h(t, u), \qquad h \in [0, \infty]$$

Vector case: $y = h(t, u), \qquad h^T = \left[\ h_1, \ \ h_2, \ \ \cdots, \ \ h_p\ \right]$

power inflow $= \Sigma_{i=1}^{p} u_i y_i = \ u^T y$

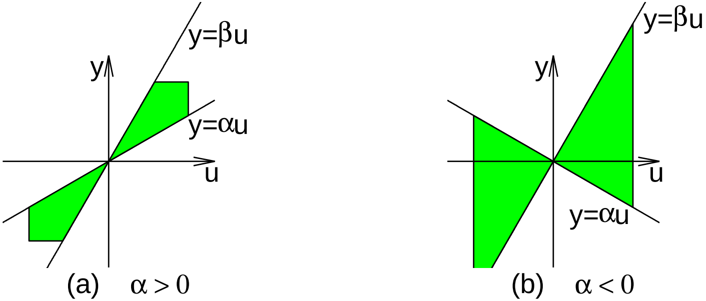
Definition 5.1

$y = h(t, u)$ is

- passive if $u^T y \geq 0$
- lossless if $u^T y = 0$
- input strictly passive if $u^T y \geq u^T \varphi(u)$ for some function φ where $u^T \varphi(u) > 0$, $\forall\ u \neq 0$
- output strictly passive if $u^T y \geq y^T \rho(y)$ for some function ρ where $y^T \rho(y) > 0$, $\forall\ y \neq 0$

Sector Nonlinearity: h belongs to the sector $[\alpha, \beta]$ ($h \in [\alpha, \beta]$) if

$$\alpha u^2 \leq u h(t, u) \leq \beta u^2$$



Also, $h \in (\alpha, \beta]$, $h \in [\alpha, \beta)$, $h \in (\alpha, \beta)$

$$\alpha u^2 \leq uh(t,u) \leq \beta u^2 \Leftrightarrow [h(t,u) - \alpha u][h(t,u) - \beta u] \leq 0$$

Definition 5.2

A memoryless function $h(t,u)$ is said to belong to the sector

- $[0, \infty]$ if $u^T h(t,u) \geq 0$
- $[K_1, \infty]$ if $u^T [h(t,u) - K_1 u] \geq 0$
- $[0, K_2]$ with $K_2 = K_2^T > 0$ if $h^T(t,u)[h(t,u) - K_2 u] \leq 0$
- $[K_1, K_2]$ with $K = K_2 - K_1 = K^T > 0$ if

$$[h(t,u) - K_1 u]^T [h(t,u) - K_2 u] \leq 0$$

Example

$$h(u) = \begin{bmatrix} h_1(u_1) \\ h_2(u_2) \end{bmatrix}, \quad h_i \in [\alpha_i, \beta_i], \quad \beta_i > \alpha_i \quad i = 1, 2$$

$$K_1 = \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix}, \qquad K_2 = \begin{bmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{bmatrix}$$

$$h \in [K_1, K_2]$$

$$K = K_2 - K_1 = \begin{bmatrix} \beta_1 - \alpha_1 & 0 \\ 0 & \beta_2 - \alpha_2 \end{bmatrix}$$

Example

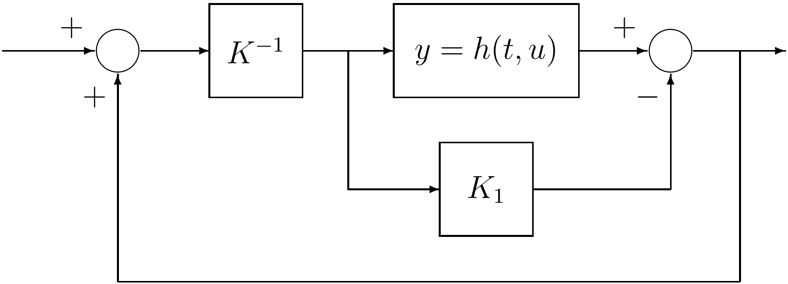
$$\|h(u) - Lu\| \le \gamma \|u\|$$

$$K_1 = L - \gamma I, \qquad K_2 = L + \gamma I$$

$$[h(u) - K_1 u]^T [h(u) - K_2 u] = \\ \|h(u) - Lu\|^2 - \gamma^2 \|u\|^2 \le 0$$

$$K = K_2 - K_1 = 2\gamma I$$

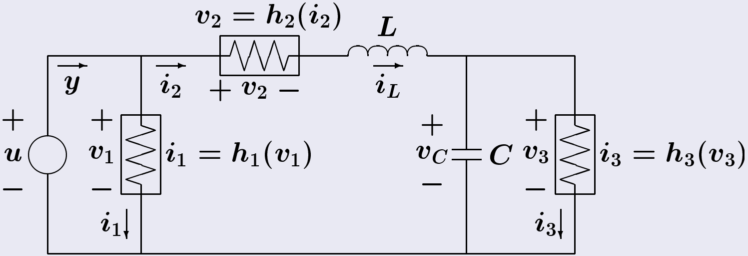
A function in the sector $[K_1, K_2]$ can be transformed into a function in the sector $[0, \infty]$ by input feedforward followed by output feedback



$$[K_1, K_2] \xrightarrow{\text{Feedforward}} [0, K] \xrightarrow{K^{-1}} [0, I] \xrightarrow{\text{Feedback}} [0, \infty]$$

State Models

Example 5.1



$$
\begin{aligned}
L\dot{x}_1 &= u - h_2(x_1) - x_2 \\
C\dot{x}_2 &= x_1 - h_3(x_2) \\
y &= x_1 + h_1(u)
\end{aligned}
$$

$$V(x) = \tfrac{1}{2}Lx_1^2 + \tfrac{1}{2}Cx_2^2$$

$$\int_0^t u(s)y(s)\ ds \geq V(x(t)) - V(x(0))$$

$$u(t)y(t) \geq \dot{V}(x(t), u(t))$$

$$\begin{aligned}
\dot{V} &= Lx_1\dot{x}_1 + Cx_2\dot{x}_2 \\
&= x_1[u - h_2(x_1) - x_2] + x_2[x_1 - h_3(x_2)] \\
&= x_1[u - h_2(x_1)] - x_2h_3(x_2) \\
&= [x_1 + h_1(u)]u - uh_1(u) - x_1h_2(x_1) - x_2h_3(x_2) \\
&= uy - uh_1(u) - x_1h_2(x_1) - x_2h_3(x_2)
\end{aligned}$$

$$uy = \dot{V} + uh_1(u) + x_1h_2(x_1) + x_2h_3(x_2)$$

If h_1, h_2, and h_3 are passive, $uy \geq \dot{V}$ and the system is passive

Case 1: If $h_1 = h_2 = h_3 = 0$, then $uy = \dot{V}$; no energy dissipation; the system is lossless

Case 2: If $h_1 \in (0, \infty]$ ($uh_1(u) > 0$ for $u \neq 0$), then

$$uy \geq \dot{V} + uh_1(u)$$

The energy absorbed over $[0, t]$ will be greater than the increase in the stored energy, unless the input $u(t)$ is identically zero. This is a case of input strict passivity

Case 3: If $h_1 = 0$ and $h_2 \in (0, \infty]$, then

$$y = x_1 \text{ and } uy \geq \dot{V} + yh_2(y)$$

The energy absorbed over $[0, t]$ will be greater than the increase in the stored energy, unless the output y is identically zero. This is a case of output strict passivity

Case 4: If $h_2 \in (0, \infty)$ and $h_3 \in (0, \infty)$, then

$$uy \geq \dot{V} + x_1 h_2(x_1) + x_2 h_3(x_2)$$

$x_1 h_2(x_1) + x_2 h_3(x_2)$ is a positive definite function of x. This is a case of state strict passivity because the energy absorbed over $[0, t]$ will be greater than the increase in the stored energy, unless the state x is identically zero

Definition 5.3

The system

$$\dot{x} = f(x, u), \qquad y = h(x, u)$$

is passive if there is a continuously differentiable positive semidefinite function $V(x)$ (the storage function) such that

$$u^T y \geq \dot{V} = \frac{\partial V}{\partial x} f(x, u), \;\; \forall \; (x, u)$$

Moreover, it is

- lossless if $u^T y = \dot{V}$
- input strictly passive if $u^T y \geq \dot{V} + u^T \varphi(u)$ for some function φ such that $u^T \varphi(u) > 0, \; \forall \; u \neq 0$
- output strictly passive if $u^T y \geq \dot{V} + y^T \rho(y)$ for some function ρ such that $y^T \rho(y) > 0, \; \forall \; y \neq 0$
- strictly passive if $u^T y \geq \dot{V} + \psi(x)$ for some positive definite function ψ

Example 5.2

$$\dot{x} = u, \qquad y = x$$

$$V(x) = \tfrac{1}{2}x^2 \quad \Rightarrow \quad uy = \dot{V} \quad \Rightarrow \quad \text{Lossless}$$

$$\dot{x} = u, \qquad y = x + h(u), \qquad h \in [0, \infty]$$

$$V(x) = \tfrac{1}{2}x^2 \quad \Rightarrow \quad uy = \dot{V} + uh(u) \quad \Rightarrow \quad \text{Passive}$$

$$h \in (0, \infty] \quad \Rightarrow \quad uh(u) > 0 \ \forall \ u \neq 0$$

$$\Rightarrow \quad \text{Input strictly passive}$$

$$\dot{x} = -h(x) + u, \qquad y = x, \qquad h \in [0, \infty]$$

$$V(x) = \tfrac{1}{2}x^2 \quad \Rightarrow \quad uy = \dot{V} + yh(y) \quad \Rightarrow \quad \text{Passive}$$

$$h \in (0, \infty] \quad \Rightarrow \quad \text{Output strictly passive}$$

Example 5.3

$$\dot{x} = u, \qquad y = h(x), \qquad h \in [0, \infty]$$

$$V(x) = \int_0^x h(\sigma)\ d\sigma \quad \Rightarrow \quad \dot{V} = h(x)\dot{x} = yu \quad \Rightarrow \quad \text{Lossless}$$

$$a\dot{x} = -x + u, \qquad y = h(x), \qquad h \in [0, \infty]$$

$$V(x) = a\int_0^x h(\sigma)\ d\sigma \quad \Rightarrow \quad \dot{V} = h(x)(-x+u) = yu - xh(x)$$

$$yu = \dot{V} + xh(x) \quad \Rightarrow \quad \text{Passive}$$

$$h \in (0, \infty] \quad \Rightarrow \quad \text{Strictly passive}$$

Example 5.4

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -h(x_1) - ax_2 + u, \qquad y = bx_2 + u$$

$$h \in [\alpha_1, \infty], \quad a > 0, \quad b > 0, \quad \alpha_1 > 0$$

$$\begin{aligned} V(x) &= \alpha \int_0^{x_1} h(\sigma)\ d\sigma + \tfrac{1}{2}\alpha x^T P x \\ &= \alpha \int_0^{x_1} h(\sigma)\ d\sigma + \tfrac{1}{2}\alpha(p_{11}x_1^2 + 2p_{12}x_1x_2 + p_{22}x_2^2) \end{aligned}$$

$$\alpha > 0, \quad p_{11} > 0, \quad p_{11}p_{22} - p_{12}^2 > 0$$

$$
\begin{aligned}
uy - \dot{V} &= u(bx_2 + u) - \alpha[h(x_1) + p_{11}x_1 + p_{12}x_2]x_2 \\
&\quad - \alpha(p_{12}x_1 + p_{22}x_2)[-h(x_1) - ax_2 + u]
\end{aligned}
$$

Take $p_{22} = 1$, $p_{11} = ap_{12}$, and $\alpha = b$ to cancel the cross product terms

$$
uy - \dot{V} \geq bp_{12}\left(\alpha_1 - \tfrac{1}{4}bp_{12}\right)x_1^2 + b(a - p_{12})x_2^2
$$

$$
p_{12} = ak, \quad 0 < k < \min\{1, 4\alpha_1/(ab)\}
$$

$$
\Rightarrow p_{11} > 0,\ p_{11}p_{22} - p_{12}^2 > 0
$$

$$
\Rightarrow\ bp_{12}\left(\alpha_1 - \tfrac{1}{4}bp_{12}\right) > 0,\ b(a - p_{12}) > 0
$$

$\Rightarrow$ Strictly passive

Example 5.5

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\sin x_1 - bx_2 + cu, \qquad y = x_2$$
$$b \geq 0, \quad c > 0$$

$$V(x) = \alpha[(1 - \cos x_1) + \tfrac{1}{2}x_2^2], \quad \alpha > 0$$

$V(x)$ is positive semidefinite but not positive definite $\forall\ x$

$$uy - \dot{V} = ux_2 - \alpha[x_2 \sin x_1 - x_2 \sin x_1 - bx_2^2 + cx_2 u]$$

$$\alpha = 1/c \quad \Rightarrow \quad uy - \dot{V} = (b/c)x_2^2 \geq 0$$

Lossless when $b = 0$

Output strictly passive when $b > 0$