

Nonlinear Control

Lecture # 13

Passivity

Positive Real Transfer Functions

Definition 5.4

An $m \times m$ proper rational transfer function matrix $G(s)$ is positive real if

- poles of all elements of $G(s)$ are in $\text{Re}[s] \leq 0$
- for all real ω for which $j\omega$ is not a pole of any element of $G(s)$, the matrix $G(j\omega) + G^T(-j\omega)$ is positive semidefinite
- any pure imaginary pole $j\omega$ of any element of $G(s)$ is a simple pole and the residue matrix $\lim_{s \rightarrow j\omega} (s - j\omega)G(s)$ is positive semidefinite Hermitian

$G(s)$ is strictly positive real if $G(s - \varepsilon)$ is positive real for some $\varepsilon > 0$

Scalar Case ($m = 1$):

$$G(j\omega) + G^T(-j\omega) = 2\operatorname{Re}[G(j\omega)]$$

$\operatorname{Re}[G(j\omega)]$ is an even function of ω . The second condition of the definition reduces to

$$\operatorname{Re}[G(j\omega)] \geq 0, \quad \forall \omega \in [0, \infty)$$

which holds when the Nyquist plot of $G(j\omega)$ lies in the closed right-half complex plane

This is true only if the relative degree of the transfer function is zero or one

Lemma 5.1

An $m \times m$ proper rational transfer function matrix $G(s)$ is strictly positive real if and only if

- $G(s)$ is Hurwitz
- $G(j\omega) + G^T(-j\omega) > 0, \forall \omega \in R$
- $G(\infty) + G^T(\infty) > 0$ or

$$\lim_{\omega \rightarrow \infty} \omega^{2(m-q)} \det[G(j\omega) + G^T(-j\omega)] > 0$$

where $q = \text{rank}[G(\infty) + G^T(\infty)]$

Scalar Case ($m = 1$): $G(s)$ is strictly positive real if and only if

- $G(s)$ is Hurwitz
- $\operatorname{Re}[G(j\omega)] > 0, \forall \omega \in [0, \infty)$
- $G(\infty) > 0$ or

$$\lim_{\omega \rightarrow \infty} \omega^2 \operatorname{Re}[G(j\omega)] > 0$$

Example 5.6

$$G(s) = \frac{1}{s}$$

has a simple pole at $s = 0$ whose residue is 1

$$\operatorname{Re}[G(j\omega)] = \operatorname{Re}\left[\frac{1}{j\omega}\right] = 0, \quad \forall \omega \neq 0$$

Hence, G is positive real. It is not strictly positive real since

$$\frac{1}{(s - \varepsilon)}$$

has a pole in $\operatorname{Re}[s] > 0$ for any $\varepsilon > 0$

$$G(s) = \frac{1}{s+a}, \quad a > 0, \quad \text{is Hurwitz}$$

$$\operatorname{Re}[G(j\omega)] = \frac{a}{\omega^2 + a^2} > 0, \quad \forall \omega \in [0, \infty)$$

$$\lim_{\omega \rightarrow \infty} \omega^2 \operatorname{Re}[G(j\omega)] = \lim_{\omega \rightarrow \infty} \frac{\omega^2 a}{\omega^2 + a^2} = a > 0 \quad \Rightarrow \quad G \text{ is SPR}$$

$$G(s) = \frac{1}{s^2 + s + 1}, \quad \operatorname{Re}[G(j\omega)] = \frac{1 - \omega^2}{(1 - \omega^2)^2 + \omega^2}$$

G is not PR

$$G(s) = \begin{bmatrix} \frac{s+2}{s+1} & \frac{1}{s+2} \\ \frac{-1}{s+2} & \frac{2}{s+1} \end{bmatrix} \text{ is Hurwitz}$$

$$G(j\omega) + G^T(-j\omega) = \begin{bmatrix} \frac{2(2+\omega^2)}{1+\omega^2} & \frac{-2j\omega}{4+\omega^2} \\ \frac{2j\omega}{4+\omega^2} & \frac{4}{1+\omega^2} \end{bmatrix} > 0, \quad \forall \omega \in R$$

$$G(\infty) + G^T(\infty) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \quad q = 1$$

$$\lim_{\omega \rightarrow \infty} \omega^2 \det[G(j\omega) + G^T(-j\omega)] = 4 \Rightarrow G \text{ is SPR}$$

Positive Real Lemma (5.2)

Let

$$G(s) = C(sI - A)^{-1}B + D$$

where (A, B) is controllable and (A, C) is observable. $G(s)$ is positive real if and only if there exist matrices $P = P^T > 0$, L , and W such that

$$\begin{aligned} PA + A^T P &= -L^T L \\ PB &= C^T - L^T W \\ W^T W &= D + D^T \end{aligned}$$

Kalman–Yakubovich–Popov Lemma (5.3)

Let

$$G(s) = C(sI - A)^{-1}B + D$$

where (A, B) is controllable and (A, C) is observable. $G(s)$ is strictly positive real if and only if there exist matrices $P = P^T > 0$, L , and W , and a positive constant ε such that

$$\begin{aligned} PA + A^T P &= -L^T L - \varepsilon P \\ PB &= C^T - L^T W \\ W^T W &= D + D^T \end{aligned}$$

Lemma 5.4

The linear time-invariant minimal realization

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

with

$$G(s) = C(sI - A)^{-1}B + D$$

is

- passive if $G(s)$ is positive real
- strictly passive if $G(s)$ is strictly positive real

Proof

Apply the PR and KYP Lemmas, respectively, and use $V(x) = \frac{1}{2}x^T Px$ as the storage function

$$\begin{aligned}
& u^T y - \frac{\partial V}{\partial x}(Ax + Bu) \\
&= u^T(Cx + Du) - x^T P(Ax + Bu) \\
&= u^T Cx + \frac{1}{2}u^T(D + D^T)u \\
&\quad - \frac{1}{2}x^T(PA + A^T P)x - x^T P Bu \\
&= u^T(B^T P + W^T L)x + \frac{1}{2}u^T W^T W u \\
&\quad + \frac{1}{2}x^T L^T L x + \frac{1}{2}\varepsilon x^T P x - x^T P Bu \\
&= \frac{1}{2}(Lx + Wu)^T(Lx + Wu) + \frac{1}{2}\varepsilon x^T P x \geq \frac{1}{2}\varepsilon x^T P x
\end{aligned}$$

In the case of the PR Lemma, $\varepsilon = 0$, and we conclude that the system is passive; in the case of the KYP Lemma, $\varepsilon > 0$, and we conclude that the system is strictly passive

Connection with Stability

Lemma 5.5

If the system

$$\dot{x} = f(x, u), \quad y = h(x, u)$$

is passive with a positive definite storage function $V(x)$, then the origin of $\dot{x} = f(x, 0)$ is stable

Proof

$$u^T y \geq \frac{\partial V}{\partial x} f(x, u) \Rightarrow \frac{\partial V}{\partial x} f(x, 0) \leq 0$$

Lemma 5.6

If the system

$$\dot{x} = f(x, u), \quad y = h(x, u)$$

is strictly passive, then the origin of $\dot{x} = f(x, 0)$ is asymptotically stable. Furthermore, if the storage function is radially unbounded, the origin will be globally asymptotically stable

Proof

The storage function $V(x)$ is positive definite

$$u^T y \geq \frac{\partial V}{\partial x} f(x, u) + \psi(x) \quad \Rightarrow \quad \frac{\partial V}{\partial x} f(x, 0) \leq -\psi(x)$$

Why is $V(x)$ positive definite? Let $\phi(t; x)$ be the solution of $\dot{z} = f(z, 0)$, $z(0) = x$

$$\dot{V} \leq -\psi(x)$$

$$V(\phi(\tau; x)) - V(x) \leq - \int_0^\tau \psi(\phi(t; x)) dt, \quad \forall \tau \in [0, \delta]$$

$$V(\phi(\tau; x)) \geq 0 \quad \Rightarrow \quad V(x) \geq \int_0^\tau \psi(\phi(t; x)) dt$$

$$V(\bar{x}) = 0 \quad \Rightarrow \quad \int_0^\tau \psi(\phi(t; \bar{x})) dt = 0, \quad \forall \tau \in [0, \delta]$$

$$\Rightarrow \psi(\phi(t; \bar{x})) \equiv 0 \quad \Rightarrow \quad \phi(t; \bar{x}) \equiv 0 \quad \Rightarrow \quad \bar{x} = 0$$

Definition 5.5

The system

$$\dot{x} = f(x, u), \quad y = h(x, u)$$

is zero-state observable if no solution of $\dot{x} = f(x, 0)$ can stay identically in $S = \{h(x, 0) = 0\}$, other than the zero solution $x(t) \equiv 0$

Linear Systems

$$\dot{x} = Ax, \quad y = Cx$$

Observability of (A, C) is equivalent to

$$y(t) = Ce^{At}x(0) \equiv 0 \Leftrightarrow x(0) = 0 \Leftrightarrow x(t) \equiv 0$$

Lemma 5.6

If the system

$$\dot{x} = f(x, u), \quad y = h(x, u)$$

is output strictly passive and zero-state observable, then the origin of $\dot{x} = f(x, 0)$ is asymptotically stable. Furthermore, if the storage function is radially unbounded, the origin will be globally asymptotically stable

Proof

The storage function $V(x)$ is positive definite

$$u^T y \geq \frac{\partial V}{\partial x} f(x, u) + y^T \rho(y) \Rightarrow \frac{\partial V}{\partial x} f(x, 0) \leq -y^T \rho(y)$$

$$\dot{V}(x(t)) \equiv 0 \Rightarrow y(t) \equiv 0 \Rightarrow x(t) \equiv 0$$

Apply the invariance principle

Example 5.7

$$\dot{x} = f(x) + G(x)u, \quad y = h(x), \quad \dim(u) = \dim(y)$$

Suppose there is $V(x)$ such that

$$\frac{\partial V}{\partial x} f(x) \leq 0, \quad \frac{\partial V}{\partial x} G(x) = h^T(x)$$

$$u^T y - \dot{V} = u^T h(x) - \frac{\partial V}{\partial x} f(x) - h^T(x)u = - \frac{\partial V}{\partial x} f(x) \geq 0$$

If $V(x)$ is positive definite, the origin of $\dot{x} = f(x)$ is stable

If we have the stronger condition

$$\frac{\partial V}{\partial x} f(x) \leq -k h^T(x) h(x), \quad \frac{\partial V}{\partial x} G(x) = h^T(x), \quad k > 0$$

$$u^T y - \dot{V} \geq k y^T y$$

The system is output strictly passive. If, in addition, it is zero-state observable, then the origin of $\dot{x} = f(x)$ is asymptotically stable

Example 5.8

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -ax_1^3 - kx_2 + u, \quad y = x_2, \quad a, k > 0$$

$$V(x) = \frac{1}{4}ax_1^4 + \frac{1}{2}x_2^2$$

$$\dot{V} = ax_1^3x_2 + x_2(-ax_1^3 - kx_2 + u) = -ky^2 + yu$$

The system is output strictly passive

$$y(t) \equiv 0 \Leftrightarrow x_2(t) \equiv 0 \Rightarrow ax_1^3(t) \equiv 0 \Rightarrow x_1(t) \equiv 0$$

The system is zero-state observable. V is radially unbounded. Hence, the origin of the unforced system is globally asymptotically stable