Nonlinear Control

Lecture # 14

Input-Output Stability

$\mathcal{L}$ Stability

Input-Output Models: $\quad y = Hu$

$u(t)$ is a piecewise continuous function of t and belongs to a linear space of signals

- The space of bounded functions: $\sup_{t\geq 0} \|u(t)\| < \infty$
- The space of square-integrable functions: $\int_0^\infty u^T(t)u(t)\ dt < \infty$

Norm of a signal $\|u\|$:

- $\|u\| \geq 0$ and $\|u\| = 0 \quad \Leftrightarrow \quad u = 0$
- $\|au\| = a\|u\|$ for any $a > 0$
- Triangle Inequality: $\|u_1 + u_2\| \leq \|u_1\| + \|u_2\|$

$\mathcal{L}_p$ spaces:

$$\mathcal{L}_\infty : \quad \|u\|_{\mathcal{L}_\infty} = \sup_{t\geq 0} \|u(t)\| < \infty$$

$$\mathcal{L}_2 : \quad \|u\|_{\mathcal{L}_2} = \sqrt{\int_0^\infty u^T(t)u(t)\ dt} < \infty$$

$$\mathcal{L}_p : \quad \|u\|_{\mathcal{L}_p} = \left(\int_0^\infty \|u(t)\|^p\ dt\right)^{1/p} < \infty, \quad 1 \leq p < \infty$$

Notation $\mathcal{L}_p^m$: p is the type of p-norm used to define the space and m is the dimension of u

Extended Space: $\mathcal{L}_e = \{u \mid u_\tau \in \mathcal{L}, \forall\ \tau \in [0,\infty)\}$

u_τ is a truncation of u: $u_\tau(t) = \begin{cases} u(t), & 0 \le t \le \tau \\ 0, & t > \tau \end{cases}$

$\mathcal{L}_e$ is a linear space and $\mathcal{L} \subset \mathcal{L}_e$

Example

$$u(t) = t, \qquad u_\tau(t) = \begin{cases} t, & 0 \le t \le \tau \\ 0, & t > \tau \end{cases}$$

$u \notin \mathcal{L}_\infty$ but $u_\tau \in \mathcal{L}_{\infty e}$

Causality: A mapping $H : \mathcal{L}_e^m \to \mathcal{L}_e^q$ is causal if the value of the output $(Hu)(t)$ at any time t depends only on the values of the input up to time t

$$(Hu)_\tau = (Hu_\tau)_\tau$$

Definition 6.1

A scalar continuous function $g(r)$, defined for $r \in [0, a)$, is a gain function if it is nondecreasing and $g(0) = 0$

A class $\mathcal{K}$ function is a gain function but not the other way around. By not requiring the gain function to be strictly increasing we can have $g = 0$ or $g(r) = \text{sat}(r)$

Definition 6.2

A mapping $H : \mathcal{L}_e^m \to \mathcal{L}_e^q$ is $\mathcal{L}$ stable if there exist a gain function g, defined on $[0, \infty)$, and a nonnegative constant β such that

$$\|(Hu)_\tau\|_{\mathcal{L}} \le g\left(\|u_\tau\|_{\mathcal{L}}\right) + \beta, \quad \forall\, u \in \mathcal{L}_e^m \text{ and } \tau \in [0, \infty)$$

It is finite-gain $\mathcal{L}$ stable if there exist nonnegative constants γ and β such that

$$\|(Hu)_\tau\|_{\mathcal{L}} \le \gamma\|u_\tau\|_{\mathcal{L}} + \beta, \quad \forall\, u \in \mathcal{L}_e^m \text{ and } \tau \in [0, \infty)$$

In this case, we say that the system has $\mathcal{L}$ gain $\le \gamma$. The bias term β is included in the definition to allow for systems where Hu does not vanish at $u = 0$.

Example 6.1: Memoryless function $y = h(u)$

Suppose $\quad |h(u)| \le a + b|u|, \;\; \forall\, u \in R$

Finite-gain $\mathcal{L}_\infty$ stable with $\beta = a$ and $\gamma = b$

If $a = 0$, then for each $p \in [1, \infty)$

$$\int_0^\infty |h(u(t))|^p \; dt \le (b)^p \int_0^\infty |u(t)|^p \; dt$$

Finite-gain $\mathcal{L}_p$ stable with $\beta = 0$ and $\gamma = b$

For $h(u) = u^2$, H is $\mathcal{L}_\infty$ stable with zero bias and $g(r) = r^2$. It is not finite-gain $\mathcal{L}_\infty$ stable because $|h(u)| = u^2$ cannot be bounded $\gamma|u| + \beta$ for all $u \in R$

Example 6.2: SISO causal convolution operator

$$y(t) = \int_0^t h(t-\sigma)u(\sigma)\ d\sigma, \qquad h(t) = 0 \ \text{ for } \ t < 0$$

Suppose $h \in \mathcal{L}_1 \quad \Leftrightarrow \quad \|h\|_{\mathcal{L}_1} = \int_0^\infty |h(\sigma)|\ d\sigma < \infty$

$$\begin{array}{rcl} |y(t)| & \leq & \int_0^t |h(t-\sigma)|\ |u(\sigma)|\ d\sigma \\ & \leq & \int_0^t |h(t-\sigma)|\ d\sigma \sup_{0\leq\sigma\leq\tau} |u(\sigma)| \\ & = & \int_0^t |h(s)|\ ds \sup_{0\leq\sigma\leq\tau} |u(\sigma)| \end{array}$$

$$\|y_\tau\|_{\mathcal{L}_\infty} \leq \|h\|_{\mathcal{L}_1} \|u_\tau\|_{\mathcal{L}_\infty}, \quad \forall\ \tau \in [0, \infty)$$

Finite-gain $\mathcal{L}_\infty$ stable

Also, finite-gain $\mathcal{L}_p$ stable for $p \in [1, \infty)$ (see textbook)

Small-signal $\mathcal{L}$ Stability

Example 6.3

$$y = \tan u$$

The output $y(t)$ is defined only when the input signal is restricted to $|u(t)| < \pi/2$ for all $t \geq 0$

$$u(t) \in \{|u| \leq r < \pi/2\} \quad \Rightarrow \quad |y| \leq \left(\frac{\tan r}{r}\right)|u|$$

$$\|y\|_{\mathcal{L}_p} \leq \left(\frac{\tan r}{r}\right)\|u\|_{\mathcal{L}_p}, \quad p \in [1, \infty]$$

Definition 6.3

A mapping $H : \mathcal{L}_e^m \to \mathcal{L}_e^q$ is small-signal $\mathcal{L}$ stable (respectively, small-signal finite-gain $\mathcal{L}$ stable) if there is a positive constant r such that the condition for $\mathcal{L}$ stability (respectively, finite-gain $\mathcal{L}$ stability) is satisfied for all $u \in \mathcal{L}_e^m$ with $\sup_{0 \le t \le \tau} \|u(t)\| \le r$

$\mathcal{L}$ Stability of State Models

$$\dot{x} = f(x,u), \quad y = h(x,u), \qquad 0 = f(0,0), \; 0 = h(0,0)$$

Case 1: The origin of $\dot{x} = f(x,0)$ is exponentially stable

Theorem 6.1

Suppose, $\forall \; \|x\| \le r$, $\forall \; \|u\| \le r_u$,

$$c_1\|x\|^2 \le V(x) \le c_2\|x\|^2$$

$$\frac{\partial V}{\partial x} f(x,0) \le -c_3\|x\|^2, \quad \left\|\frac{\partial V}{\partial x}\right\| \le c_4\|x\|$$

$$\|f(x,u) - f(x,0)\| \le L\|u\|, \quad \|h(x,u)\| \le \eta_1\|x\| + \eta_2\|u\|$$

Then, for each x_0 with $\|x_0\| \le r\sqrt{c_1/c_2}$, the system is small-signal finite-gain $\mathcal{L}_p$ stable for each $p \in [1,\infty]$. It is finite-gain $\mathcal{L}_p$ stable $\forall \; x_0 \in R^n$ if the assumptions hold globally [see the textbook for β and γ]

Proof

$$\dot{V} = \frac{\partial V}{\partial x} f(x,0) + \frac{\partial V}{\partial x}[f(x,u) - f(x,0)]$$

$$\dot{V} \le -c_3\|x\|^2 + c_4 L\|x\| \; \|u\| \;\le\; -\frac{c_3}{c_2}V + \frac{c_4 L}{\sqrt{c_1}}\|u\|\sqrt{V}$$

$$W(x) = \sqrt{V(x)}$$

$$\dot{W} \le -aW + b\|u(t)\|, \quad a = \frac{c_3}{2c_2},\; b = \frac{c_4 L}{2\sqrt{c_1}}$$

$$U(t) = e^{at}W(x(t)) \;\Rightarrow\; \dot{U} = e^{at}\dot{W} + ae^{at}W \le be^{at}\|u\|$$

$$U(t) \le U(0) + \int_0^t be^{a\tau}\|u(\tau)\| \; d\tau$$

$$W(x(t)) \le e^{-at}W(x(0)) + \int_0^t e^{-a(t-\tau)} b\|u(\tau)\| \, d\tau$$

$$\sqrt{c_1}\|x\| \le W(x) \le \sqrt{c_2}\|x\|$$

$$\|x(t)\| \le \sqrt{\frac{c_2}{c_1}}\|x(0)\|e^{-at} + \frac{c_4 L}{2c_1}\int_0^t e^{-at}\|u(\tau)\| \, d\tau$$

$$\|y(t)\| \le \eta_1\|x(t)\| + \eta_2\|u(t)\|$$

$$\|y(t)\| \le k_0\|x(0)\|e^{-at} + k_2\int_0^t e^{-a(t-\tau)}\|u(\tau)\| \, d\tau + k_3 \ \|u(t)\|$$

Example 6.4

$$\dot{x} = -x - x^3 + u, \qquad y = \tanh x + u$$

$$V = \frac{1}{2}x^2 \quad \Rightarrow \quad \dot{V} = x(-x - x^3) \le -x^2$$

$$c_1 = c_2 = \frac{1}{2}, \; c_3 = c_4 = 1, \quad L = \eta_1 = \eta_2 = 1$$

Finite-gain $\mathcal{L}_p$ stable for each $x(0) \in R$ and each $p \in [1, \infty]$

Example 6.5

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 - x_2 - a\tanh x_1 + u, \quad y = x_1, \quad a \ge 0$$

$$V(x) = x^T P x = p_{11}x_1^2 + 2p_{12}x_1x_2 + p_{22}x_2^2$$

$$\dot{V} = -2p_{12}(x_1^2 + ax_1 \tanh x_1) + 2(p_{11} - p_{12} - p_{22})x_1x_2 - 2ap_{22}x_2 \tanh x_1 - 2(p_{22} - p_{12})x_2^2$$

$$p_{11} = p_{12} + p_{22} \quad \Rightarrow \quad \text{the term } x_1x_2 \text{ is canceled}$$

$$p_{22} = 2p_{12} = 1 \quad \Rightarrow \quad P \text{ is positive definite}$$

$$\dot{V} = -x_1^2 - x_2^2 - ax_1 \tanh x_1 - 2ax_2 \tanh x_1$$

$$\dot{V} \le -\|x\|^2 + 2a|x_1|\ |x_2| \le -(1-a)\|x\|^2$$

$$a < 1 \Rightarrow c_1 = \lambda_{\min}(P), c_2 = \lambda_{\max}(P), c_3 = 1 - a, c_4 = 2c_2$$

$$L = \eta_1 = 1, \quad \eta_2 = 0$$

For each $x(0) \in R^2,\ p \in [1, \infty]$, the system is finite-gain $\mathcal{L}_p$ stable

$$\gamma = 2[\lambda_{\max}(P)]^2 / [(1-a)\lambda_{\min}(P)]$$

Case 2: The origin of $\dot{x} = f(x, 0)$ is asymptotically stable

Theorem 6.2

Suppose that, for all (x, u), f is locally Lipschitz and h is continuous and satisfies

$$\|h(x, u)\| \le g_1(\|x\|) + g_2(\|u\|) + \eta, \quad \eta \ge 0$$

for some gain functions g_1, g_2. If $\dot{x} = f(x, u)$ is ISS, then, for each $x(0) \in R^n$, the system

$$\dot{x} = f(x, u), \qquad y = h(x, u)$$

is $\mathcal{L}_\infty$ stable

Proof

$$\|x(t)\| \le \max\left\{\beta(\|x_0\|,t), \gamma\left(\sup_{0\le t\le\tau}\|u(t)\|\right)\right\}$$

$$\begin{aligned}\|y(t)\| \;&\le\; g_1\left(\max\left\{\beta(\|x_0\|,t), \gamma\left(\sup_{0\le t\le\tau}\|u(t)\|\right)\right\}\right)\\ &\quad+g_2(\|u(t)\|)+\eta\end{aligned}$$

$$g_1(\max\{a,b\}) \le g_1(a) + g_1(b)$$

$$\|y_\tau\|_{\mathcal{L}_\infty} \le g\left(\|u_\tau\|_{\mathcal{L}_\infty}\right) + \beta_0$$

$$g = g_1 \circ \gamma + g_2 \quad \text{and} \quad \beta_0 = g_1(\beta(\|x_0\|,0)) + \eta$$

Theorem 6.3

Suppose f is locally Lipschitz and h is continuous in some neighborhood of $(x = 0,\ u = 0)$. If the origin of $\dot{x} = f(x, 0)$ is asymptotically stable, then there is a constant $k_1 > 0$ such that for each $x(0)$ with $\|x(0)\| < k_1$, the system

$$\dot{x} = f(x, u), \qquad y = h(x, u)$$

is small-signal $\mathcal{L}_\infty$ stable

Proof

Use Lemma 4.7 (asymptotic stability is equivalent to local ISS)

Example 6.6

$$\dot{x} = -x - 2x^3 + (1 + x^2)u^2, \qquad y = x^2 + u$$

ISS from Example 4.13

$$g_1(r) = r^2, \quad g_2(r) = r, \quad \eta = 0$$

$\mathcal{L}_\infty$ stable

Example 6.7

$$\dot{x}_1 = -x_1^3 + x_2, \quad \dot{x}_2 = -x_1 - x_2^3 + u, \qquad y = x_1 + x_2$$

$$V = (x_1^2 + x_2^2) \quad \Rightarrow \quad \dot{V} = -2x_1^4 - 2x_2^4 + 2x_2 u$$

$$x_1^4 + x_2^4 \geq \tfrac{1}{2}\|x\|^4$$

$$\begin{array}{rcl} \dot{V} & \leq & -\|x\|^4 + 2\|x\||u| \\ & = & -(1-\theta)\|x\|^4 - \theta\|x\|^4 + 2\|x\||u|, \quad 0 < \theta < 1 \\ & \leq & -(1-\theta)\|x\|^4, \quad \forall\ \|x\| \geq \left(\frac{2|u|}{\theta}\right)^{1/3} \ \Rightarrow \ \text{ISS} \end{array}$$

$$g_1(r) = \sqrt{2}r, \quad g_2 = 0, \quad \eta = 0$$

$\mathcal{L}_\infty$ stable