Nonlinear Control

Lecture # 15

Input-Output Stability

$\mathcal{L}_2$ Gain

Theorem 6.4

Consider the linear time-invariant system

$$\dot{x} = Ax + Bu, \qquad y = Cx + Du$$

where A is Hurwitz. Let $G(s) = C(sI - A)^{-1}B + D$

The $\mathcal{L}_2$ gain $\leq \sup_{\omega \in R} \|G(j\omega)\|$

Actually, $\mathcal{L}_2$ gain $= \sup_{\omega \in R} \|G(j\omega)\|$

Proof

$$U(j\omega) = \int_0^\infty u(t)e^{-j\omega t}\;dt, \quad Y(j\omega) = G(j\omega)U(j\omega)$$

By Parseval's theorem

$$\begin{aligned}
\|y\|_{\mathcal{L}_2}^2 &= \int_0^\infty y^T(t)y(t)\;dt \;=\; \frac{1}{2\pi}\int_{-\infty}^\infty Y^*(j\omega)Y(j\omega)\;d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^\infty U^*(j\omega)G^T(-j\omega)G(j\omega)U(j\omega)\;d\omega \\
&\le \left(\sup_{\omega\in R}\|G(j\omega)\|\right)^2 \frac{1}{2\pi}\int_{-\infty}^\infty U^*(j\omega)U(j\omega)\;d\omega \\
&= \left(\sup_{\omega\in R}\|G(j\omega)\|\right)^2 \|u\|_{\mathcal{L}_2}^2
\end{aligned}$$

Lemma 6.1

Consider the time-invariant system

$$\dot{x} = f(x, u), \qquad y = h(x, u)$$

where f is locally Lipschitz and h is continuous for all $x \in R^n$ and $u \in R^m$. Let $V(x)$ be a positive semidefinite function such that

$$\dot{V} = \frac{\partial V}{\partial x} f(x, u) \le k(\gamma^2 \|u\|^2 - \|y\|^2), \qquad k, \gamma > 0$$

Then, for each $x(0) \in R^n$, the system is finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain is less than or equal to γ. In particular

$$\|y_\tau\|_{\mathcal{L}_2} \le \gamma \|u_\tau\|_{\mathcal{L}_2} + \sqrt{\frac{V(x(0))}{k}}$$

Proof

$$V(x(\tau)) - V(x(0)) \leq k\gamma^2 \int_0^\tau \|u(t)\|^2 \, dt - \ k \int_0^\tau \|y(t)\|^2 \, dt$$

$$V(x) \geq 0$$

$$\int_0^\tau \|y(t)\|^2 \, dt \leq \gamma^2 \int_0^\tau \|u(t)\|^2 \, dt + \frac{V(x(0))}{k}$$

$$\|y_\tau\|_{\mathcal{L}_2} \leq \gamma \|u_\tau\|_{\mathcal{L}_2} + \sqrt{\frac{V(x(0))}{k}}$$

Theorem 6.5

If the system

$$\dot{x} = f(x,u), \qquad y = h(x,u)$$

is output strictly passive with

$$u^T y \geq \dot{V} + \delta y^T y, \quad \delta > 0$$

then it is finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain is less than or equal to $1/\delta$

Proof

$$\begin{aligned} \dot{V} &\leq u^T y - \delta y^T y \\ &= -\tfrac{1}{2\delta}(u - \delta y)^T (u - \delta y) + \tfrac{1}{2\delta} u^T u - \tfrac{\delta}{2} y^T y \\ &\leq \tfrac{\delta}{2}\left(\tfrac{1}{\delta^2} u^T u - y^T y\right) \end{aligned}$$

Theorem 6.6

Consider the time-invariant system

$$\dot{x} = f(x) + G(x)u, \qquad y = h(x)$$

$$f(0) = 0, \qquad h(0) = 0$$

where f and G are locally Lipschitz and h is continuous over R^n. Suppose $\exists\ \gamma > 0$ and a continuously differentiable, positive semidefinite function $V(x)$ that satisfies the Hamilton–Jacobi inequality

$$\frac{\partial V}{\partial x} f(x) + \frac{1}{2\gamma^2} \frac{\partial V}{\partial x} G(x) G^T(x) \left(\frac{\partial V}{\partial x} \right)^T + \frac{1}{2} h^T(x) h(x) \leq 0$$

$\forall\ x \in R^n$. Then, for each $x(0) \in R^n$, the system is finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain $\leq \gamma$

Proof

$$
\begin{aligned}
\frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} G(x) u = \\
& -\frac{1}{2}\gamma^2 \left\| u - \frac{1}{\gamma^2} G^T(x) \left(\frac{\partial V}{\partial x}\right)^T \right\|^2 + \frac{\partial V}{\partial x} f(x) \\
& + \frac{1}{2\gamma^2} \frac{\partial V}{\partial x} G(x) G^T(x) \left(\frac{\partial V}{\partial x}\right)^T + \frac{1}{2}\gamma^2 \|u\|^2
\end{aligned}
$$

$$
\dot{V} \le \frac{1}{2}\gamma^2 \|u\|^2 - \frac{1}{2}\|y\|^2
$$

Example 6.8

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -ax_1^3 - kx_2 + u, \quad y = x_2, \quad a, k > 0$$

$$V(x) = \tfrac{a}{4}x_1^4 + \tfrac{1}{2}x_2^2$$

$$\begin{aligned} \dot{V} &= ax_1^3 x_2 + x_2(-ax_1^3 - kx_2 + u) \\ &= -kx_2^2 + x_2 u \; = \; -ky^2 + yu \end{aligned}$$

The system is finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain is less than or equal to $1/k$

Example 6.9

$$\dot{x} = Ax + Bu, \qquad y = Cx$$

Suppose there is $P = P^T \geq 0$ that satisfies the Riccati equation

$$PA + A^T P + \frac{1}{\gamma^2} PBB^T P + C^T C = 0$$

for some $\gamma > 0$. Verify that $V(x) = \frac{1}{2}x^T Px$ satisfies the Hamilton-Jacobi equation

The system is finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain is less than or equal to γ

Local Versions

Lemma 6.2

Suppose $V(x)$ satisfies

$$\dot{V} = \frac{\partial V}{\partial x} f(x,u) \leq k(\gamma^2 \|u\|^2 - \|y\|^2), \qquad k, \gamma > 0$$

for $x \in D \subset R^n$ and $u \in D_u \subset R^m$, where D and D_u are domains that contain $x = 0$ and $u = 0$, respectively. Suppose further that $x = 0$ is an asymptotically stable equilibrium point of $\dot{x} = f(x, 0)$. Then, there is $r > 0$ such that for each $x(0)$ with $\|x(0)\| \leq r$, the system

$$\dot{x} = f(x,u), \qquad y = h(x,u)$$

is small-signal finite-gain $\mathcal{L}_2$ stable with $\mathcal{L}_2$ gain less than or equal to γ

Theorem 6.7

Consider the system

$$\dot{x} = f(x,u), \qquad y = h(x,u)$$

Assume

$$u^T y \geq \dot{V} + \delta y^T y, \quad \delta > 0$$

is satisfied for $V(x) \geq 0$ in some neighborhood of $(x = 0, u = 0)$ and the origin is an asymptotically stable equilibrium point of $\dot{x} = f(x,0)$. Then, the system is small-signal finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain is less than or equal to $1/\delta$

Theorem 6.8

Consider the system

$$\dot{x} = f(x) + G(x)u, \qquad y = h(x)$$

Assume

$$\frac{\partial V}{\partial x} f(x) + \frac{1}{2\gamma^2} \frac{\partial V}{\partial x} G(x) G^T(x) \left(\frac{\partial V}{\partial x} \right)^T + \frac{1}{2} h^T(x) h(x) \leq 0$$

is satisfied for $V(x) \geq 0$ in some neighborhood of $(x = 0, u = 0)$ and the origin is an asymptotically stable equilibrium point of $\dot{x} = f(x)$. Then, the system is small-signal finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain is less than or equal to γ

Example 6.10

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -a(x_1 - \tfrac{1}{3}x_1^3) - kx_2 + u, \quad y = x_2, \quad a,\ k > 0$$

$$V(x) = a\left(\frac{1}{2}x_1^2 - \frac{1}{12}x_1^4\right) + \frac{1}{2}x_2^2 \geq 0 \text{ for } |x_1| \leq \sqrt{6}$$

$$\dot{V} = -kx_2^2 + x_2 u = -ky^2 + yu$$

$$u = 0 \ \Rightarrow\ \dot{V} = -kx_2^2 \leq 0$$

$$x_2(t) \equiv 0 \ \Rightarrow\ x_1(t)[3 - x_1^2(t)] \equiv 0 \ \Rightarrow\ x_1(t) \equiv 0 \text{ for } |x_1| < \sqrt{3}$$

By the invariance principle, the origin is asymptotically stable when $u = 0$. By Theorem 6.7, the system is small-signal finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain is $\leq 1/k$