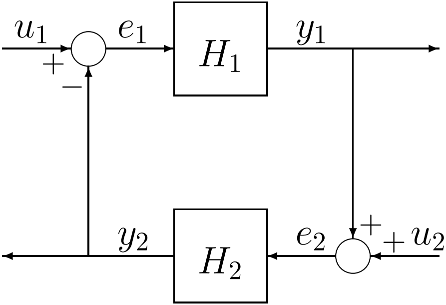
Nonlinear Control

Lecture # 16

Stability of Feedback Systems



$$\dot{x}_i = f_i(x_i, e_i), \qquad y_i = h_i(x_i, e_i)$$

$$y_i = h_i(t, e_i)$$

Passivity Theorems

Theorem 7.1

The feedback connection of two passive systems is passive

Proof

Let $V_1(x_1)$ and $V_2(x_2)$ be the storage functions for H_1 and H_2 ($V_i = 0$ if H_i is memoryless)

$$e_i^T y_i \geq \dot{V}_i, \qquad V(x) = V_1(x_1) + V_2(x_2)$$

$$e_1^T y_1 + e_2^T y_2 = (u_1 - y_2)^T y_1 + (u_2 + y_1)^T y_2 = u_1^T y_1 + u_2^T y_2$$

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \qquad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$u^T y = u_1^T y_1 + u_2^T y_2 \geq \dot{V}_1 + \dot{V}_2 = \dot{V}$$

Asymptotic Stability

Theorem 7.2

Consider the feedback connection of two dynamical systems. When $u = 0$, the origin of the closed-loop system is asymptotically stable if one of the following conditions is satisfied:

- both feedback components are strictly passive;
- both feedback components are output strictly passive and zero-state observable;
- one component is strictly passive and the other one is output strictly passive and zero-state observable.

If the storage function for each component is radially unbounded, the origin is globally asymptotically stable

Proof

H_1 is SP; H_2 is OSP & ZSO

$$e_1^T y_1 \geq \dot{V}_1 + \psi_1(x_1), \qquad \psi_1(x_1) > 0, \ \forall \ x_1 \neq 0$$

$$e_2^T y_2 \geq \dot{V}_2 + y_2^T \rho_2(y_2), \qquad y_2^T \rho(y_2) > 0, \ \forall y_2 \neq 0$$

$$e_1^T y_1 + e_2^T y_2 = (u_1 - y_2)^T y_1 + (u_2 + y_1)^T y_2 = u_1^T y_1 + u_2^T y_2$$

$$V(x) = V_1(x_1) + V_2(x_2)$$

$$\dot{V} \leq u^T y - \psi_1(x_1) - y_2^T \rho_2(y_2)$$

$$u = 0 \quad \Rightarrow \quad \dot{V} \leq -\psi_1(x_1) - y_2^T \rho_2(y_2)$$

$$\dot{V} \le -\psi_1(x_1) - y_2^T \rho_2(y_2)$$

$$\dot{V} = 0 \quad \Rightarrow \quad x_1 = 0 \text{ and } y_2 = 0$$

$$y_2(t) \equiv 0 \;\Rightarrow\; e_1(t) \equiv 0 \;\; (\;\&\; x_1(t) \equiv 0) \;\Rightarrow\; y_1(t) \equiv 0$$

$$y_1(t) \equiv 0 \;\Rightarrow\; e_2(t) \equiv 0$$

By zero-state observability of H_2: $y_2(t) \equiv 0 \;\Rightarrow\; x_2(t) \equiv 0$

Apply the invariance principle

Example 7.1

$$\underbrace{\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -ax_1^3 - kx_2 + e_1 \\ y_1 &= x_2 \end{aligned}}_{H_1} \qquad \Bigg| \qquad \underbrace{\begin{aligned} \dot{x}_3 &= x_4 \\ \dot{x}_4 &= -bx_3 - x_4^3 + e_2 \\ y_2 &= x_4 \end{aligned}}_{H_2}$$

$$a, \ b, \ k > 0$$

$$V_1 = \tfrac{1}{4}ax_1^4 + \tfrac{1}{2}x_2^2$$

$$\dot{V}_1 = ax_1^3x_2 - ax_1^3x_2 - kx_2^2 + x_2e_1 = -ky_1^2 + y_1e_1$$

With $e_1 = 0, \quad y_1(t) \equiv 0 \ \Leftrightarrow \ x_2(t) \equiv 0 \ \Rightarrow \ x_1(t) \equiv 0$

H_1 is output strictly passive and zero-state observable

$$V_2 = \frac{1}{2}bx_3^2 + \frac{1}{2}x_4^2$$

$$\dot{V}_2 = bx_3x_4 - bx_3x_4 - x_4^4 + x_4e_2 = -y_2^4 + y_2e_2$$

With $e_2 = 0, \quad y_2(t) \equiv 0 \;\Leftrightarrow\; x_4(t) \equiv 0 \;\Rightarrow\; x_3(t) \equiv 0$

H_2 is output strictly passive and zero-state observable

V_1 and V_2 are radially unbounded

The origin is globally asymptotically stable

Example 7.2

Reconsider the previous example, but change the output of H_1 to

$$y_1 = x_2 + e_1$$

$$\dot{V}_1 = -kx_2^2 + x_2 e_1 = -k(y_1 - e_1)^2 - e_1^2 + y_1 e_1$$

H_1 is passive, but we cannot conclude strict passivity or output strict passivity. We cannot apply Theorem 7.2

$$V = V_1 + V_2 = \tfrac{1}{4}ax_1^4 + \tfrac{1}{2}x_2^2 + \tfrac{1}{2}bx_3^2 + \tfrac{1}{2}x_4^2$$

$$\begin{aligned} \dot{V} &= -kx_2^2 + x_2 e_1 - x_4^4 + x_4 e_2 \\ &= -kx_2^2 - x_2 x_4 - x_4^4 + x_4(x_2 - x_4) \\ &= -kx_2^2 - x_4^4 - x_4^2 \ \le \ 0 \end{aligned}$$

$$\dot{V} = -kx_2^2 - x_4^4 - x_4^2 \ \leq \ 0$$

$$\dot{V} = 0 \ \Rightarrow \ x_2 = x_4 = 0$$

$$x_2(t) \equiv 0 \ \Rightarrow \ ax_1^3(t) - x_4(t) \equiv 0 \ \Rightarrow \ x_1(t) \equiv 0$$

$$x_4(t) \equiv 0 \ \Rightarrow \ -bx_3(t) + x_2(t) - x_4(t) \equiv 0 \ \Rightarrow \ x_3(t) \equiv 0$$

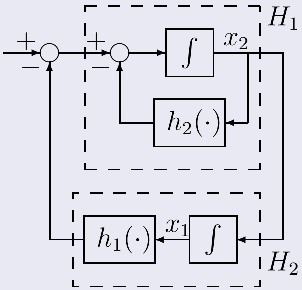
By the invariance principle and the fact that V is radially unbounded, we conclude that the origin is globally asymptotically stable

Example 7.3

Reconsider

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -h_1(x_1) - h_2(x_2), \quad h_i \in (0, \infty)$$

from Example 3.8.



$$H_1 : \qquad \dot{x}_2 = -h_2(x_2) + e_1, \quad y_1 = x_2$$

$$V_1 = \frac{1}{2}x_2^2, \qquad \dot{V}_1 = -x_2 h_2(x_2) + x_2 e_1 = -y_1 h_1(y_1) + y_1 e_1$$

Output Strictly Passive (Strictly Passive)

$$H_2 : \qquad \dot{x}_1 = e_2, \quad y_2 = h_1(x_1)$$

$$V_2 = \int_0^{x_1} h_1(\sigma)\ d\sigma, \qquad \dot{V}_2 = h_1(x_1)e_2 = y_2 e_2 \quad \text{Lossless}$$

We cannot apply Theorem 7.2, but use

$$V = V_1 + V_2 = \int_0^{x_1} h_1(\sigma)\ d\sigma + \frac{1}{2}x_2^2$$

as a Lyapunov function candidate (Examples 3.8 and 3.9)

Theorem 7.3

Consider the feedback connection of a strictly passive dynamical system with a passive time-varying memoryless function. When $u = 0$, the origin of the closed-loop system is uniformly asymptotically stable. if the storage function for the dynamical system is radially unbounded, the origin will be globally uniformly asymptotically stable

Proof

Let $V_1(x_1)$ be (positive definite) storage function of H_1.

$$\dot{V}_1 = \frac{\partial V_1}{\partial x_1} f_1(x_1, e_1) \leq e_1^T y_1 - \psi_1(x_1) = -e_2^T y_2 - \psi_1(x_1)$$

$$e_2^T y_2 \geq 0 \quad \Rightarrow \quad \dot{V}_1 \leq -\psi_1(x_1)$$

Example 7.4

Consider the feedback connection of a strictly positive real transfer function and a passive time-varying memoryless function

From Lemma 5.4, we know that the dynamical system is strictly passive with a positive definite storage function $V(x) = \frac{1}{2}x^T P x$

From Theorem 7.3, the origin of the closed-loop system is globally uniformly asymptotically stable

Theorem 7.4

Consider the feedback connection of a time-invariant dynamical system H_1 with a time-invariant memoryless function H_2. Suppose H_1 is zero-state observable, $V_1(x_1)$ is positive definite

$$e_1^T y_1 \geq \dot{V}_1 + y_1^T \rho_1(y_1), \qquad e_2^T y_2 \geq e_2^T \varphi_2(e_2)$$

Then, the origin of the closed-loop system (when $u = 0$) is asymptotically stable if

$$v^T [\rho_1(v) + \varphi_2(v)] > 0, \quad \forall\ v \neq 0$$

Furthermore, if V_1 is radially unbounded, the origin will be globally asymptotically stable

Example 7.5

$$\underbrace{\begin{array}{rcl} \dot{x} & = & f(x) + G(x)e_1 \\ y_1 & = & h(x) \end{array}}_{H_1} \quad \Bigg| \quad \underbrace{y_2 = \sigma(e_2)}_{H_2}$$

$$\sigma(0) = 0, \quad e_2^T \sigma(e_2) > 0, \ \forall \ e_2 \neq 0$$

Suppose H_1 is zero-state observable and there is a radially unbounded positive definite function $V_1(x)$ such that

$$\frac{\partial V_1}{\partial x} f(x) \leq 0, \qquad \frac{\partial V_1}{\partial x} G(x) = h^T(x), \quad \forall \ x \in R^n$$

$$\dot{V}_1 = \frac{\partial V_1}{\partial x} f(x) + \frac{\partial V_1}{\partial x} G(x) e_1 \leq y_1^T e_1$$

Apply Theorem 7.4:

$$\dot{V}_1 \leq e_1^T y_1$$

$$e_1^T y_1 \geq \dot{V}_1 + y_1^T \rho_1(y_1) \quad \text{is satisfied with} \quad \rho_1 = 0$$

$$e_2^T y_2 = e_2^T \sigma(e_2)$$

$$e_2^T y_2 \geq e_2^T \varphi_2(e_2) \quad \text{is satisfied with} \quad \varphi_2 = \sigma$$

$$v^T[\rho_1(v) + \varphi_2(v)] = v^T \sigma(v) > 0, \quad \forall\ v \neq 0$$

The origin is globally asymptotically stable