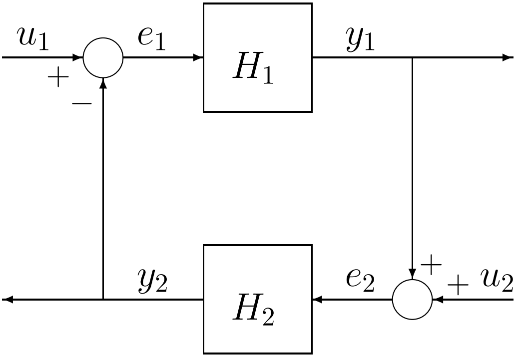
Nonlinear Control

Lecture # 17

Stability of Feedback Systems

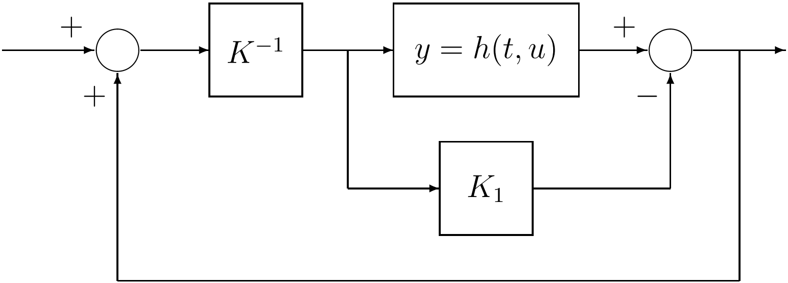


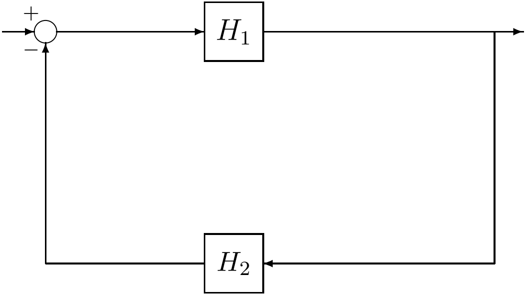
$$\dot{x}_i = f_i(x_i, e_i), \qquad y_i = h_i(x_i, e_i)$$

$$y_i = h_i(t, e_i)$$

Loop Transformations

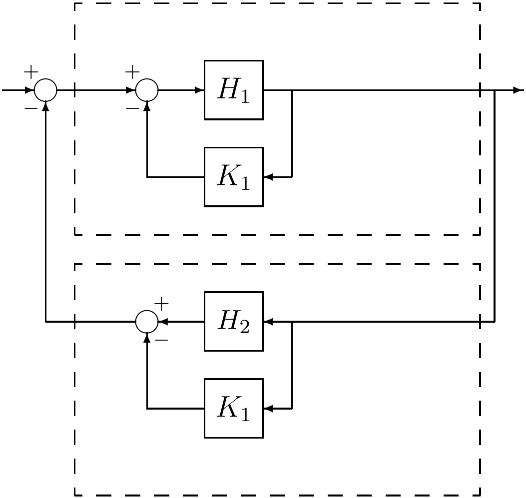
Recall that a memoryless function in the sector $[K_1, K_2]$ can be transformed into a function in the sector $[0, \infty]$ by input feedforward followed by output feedback

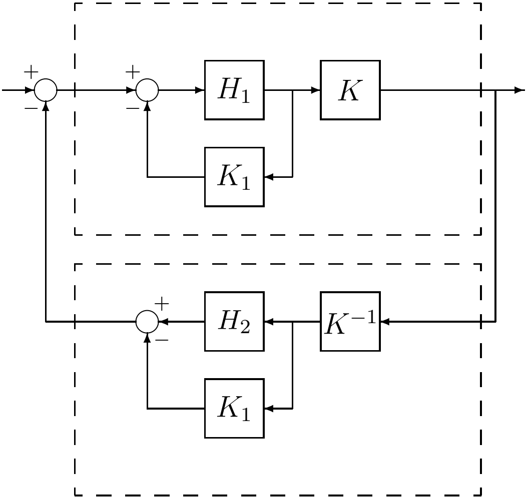


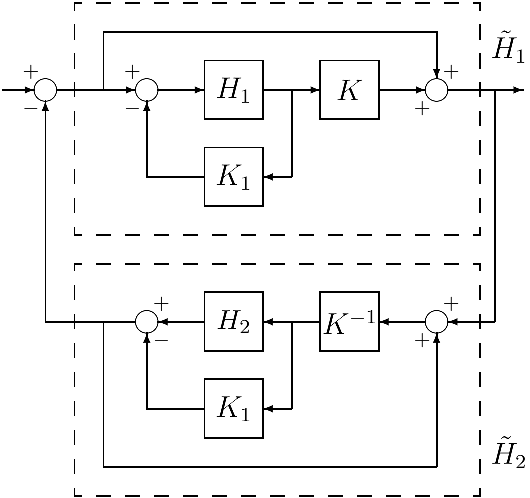


H_1 is a dynamical system

H_2 is a memoryless function in the sector $[K_1, K_2]$


+
−
+
−
H_1
K_1
+
−
H_2
K_1


+
−
+
−
H_1
K
K_1
+
−
H_2
K^{-1}
K_1



+
−
+
−
H_1
K
+
+
K_1
$\tilde{H}_1$
H_2
K^{-1}
+
−
+
+
K_1
$\tilde{H}_2$

Example 7.6

$$\underbrace{\begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -h(x_1) + cx_2 + e_1 \\ y_1 & = & x_2 \end{array}}_{H_1} \quad \Bigg| \quad \underbrace{y_2 = \sigma(e_2)}_{H_2}$$

$$\sigma \in [\alpha, \beta], \quad h \in [\alpha_1, \infty], \quad c > 0, \quad \alpha_1 > 0, \quad b = \beta - \alpha > 0$$

$$\underbrace{\begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -h(x_1) - ax_2 + \tilde{e}_1 \\ \tilde{y}_1 & = & bx_2 + \tilde{e}_1 \end{array}}_{\tilde{H}_1} \quad \Bigg| \quad \underbrace{\tilde{y}_2 = \tilde{\sigma}(\tilde{e}_2)}_{\tilde{H}_2}$$

$$\tilde{\sigma} \in [0, \infty], \qquad a = \alpha - c$$

It is shown in Example 5.4 that when $a = \alpha - c > 0$, $\tilde{H}_1$ is strictly passive with a radially unbounded storage function. Thus, we conclude from Theorem 7.3 that the origin of the feedback connection is globally asymptotically stable (when $u = 0$)

Finite-Gain $\mathcal{L}_2$ Stability

Theorem 7.5

The feedback connection of two output strictly passive systems with

$$e_i^T y_i \geq \dot{V}_i + \delta_i y_i^T y_i, \qquad \delta_i > 0$$

is finite-gain $\mathcal{L}_2$ stable and its $\mathcal{L}_2$ gain is less than or equal to $1/\min\{\delta_1, \delta_2\}$

Proof

$V = V_1 + V_2$, $\delta = \min\{\delta_1, \delta_2\}$

$$\begin{aligned} u^T y &= e_1^T y_1 + e_2^T y_2 \ \geq \ \dot{V}_1 + \delta_1 y_1^T y_1 + \dot{V}_2 + \delta_2 y_2^T y_2 \\ &\geq \dot{V} + \delta(y_1^T y_1 + y_2^T y_2) = \dot{V} + \delta y^T y \end{aligned}$$

Apply Theorem 6.5

Theorem 7.6

Consider a feedback connection where

$$e_i^T y_i \geq \dot{V}_i + \varepsilon_i e_i^T e_i + \delta_i y_i^T y_i, \quad \text{for } i = 1, 2$$

Then, the closed-loop map from u to y is finite-gain $\mathcal{L}_2$ stable if

$$\varepsilon_1 + \delta_2 > 0 \quad \text{and} \quad \varepsilon_2 + \delta_1 > 0$$

Special cases:

$\varepsilon_1 = \varepsilon_2 = 0, \delta_1 > 0, \delta_2 > 0$ Both are OSP

$\varepsilon_1 > 0, \varepsilon_2 > 0, \delta_1 = \delta_2 = 0$ Both are ISP

$\varepsilon_1 = \delta_1 = 0, \varepsilon_2 > 0, \delta_2 > 0$ H1 P, H2 OSP & ISP

$\varepsilon_1 < 0$ can be compensated for by $\delta_2 > 0$

Proof

$V(x) = V_1(x_1) + V_2(x_2)$

$$e_1^T y_1 + e_2^T y_2 = u_1^T y_1 + u_2^T y_2$$

$$e_1^T e_1 = u_1^T u_1 - 2u_1^T y_2 + y_2^T y_2, \quad e_2^T e_2 = u_2^T u_2 + 2u_2^T y_1 + y_1^T y_1$$

$$\dot{V} \le -y^T L y - u^T M u + u^T N y, \quad L = \begin{bmatrix} (\varepsilon_2 + \delta_1)I & 0 \\ 0 & (\varepsilon_1 + \delta_2)I \end{bmatrix}$$

$$a = \min\{\varepsilon_2 + \delta_1, \varepsilon_1 + \delta_2\}, \quad b = \|N\|, \quad c = \|M\|$$

$$\begin{aligned} \dot{V} &\le -a\|y\|^2 + b\|u\|\|y\| + c\|u\|^2 \\ &= -\frac{1}{2a}(b\|u\| - a\|y\|)^2 + \frac{b^2}{2a}\|u\|^2 - \frac{a}{2}\|y\|^2 + c\|u\|^2 \\ &\le \frac{k^2}{2a}\|u\|^2 - \frac{a}{2}\|y\|^2 \quad \text{Apply Lemma 6.1} \end{aligned}$$

Example 7.7

$$\underbrace{\dot{x} = f(x) + G(x)e_1, \;\; y_1 = h(x)}_{H_1} \quad \Bigg| \quad \underbrace{y_2 = \sigma(e_2)}_{H_2}$$

$$\sigma_i \in [\alpha, \beta], \quad \beta > \alpha > 0$$

$$\frac{\partial V_1}{\partial x} f(x) \le 0, \quad \frac{\partial V_1}{\partial x} G(x) = h^T(x) \;\Rightarrow\; e_1^T y_1 \ge \dot{V}_1$$

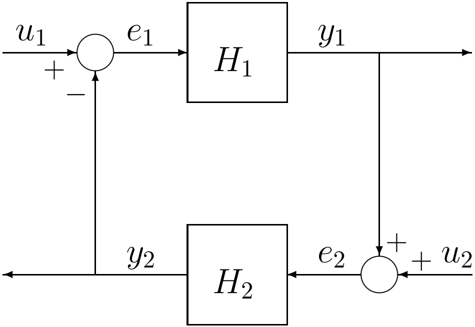
$$\alpha e_2^T e_2 \le e_2^T y_2 \le \beta e_2^T e_2 \quad \text{and} \quad \frac{1}{\beta} y_2^T y_2 \le e_2^T y_2 \le \frac{1}{\alpha} y_2^T y_2$$

$$e_2^T y_2 = \gamma e_2^T y_2 + (1-\gamma) e_2^T y_2 \ge \gamma\alpha e_2^T e_2 + (1-\gamma)\frac{1}{\beta} y_2^T y_2, \; 0 < \gamma < 1$$

$$\varepsilon_1 = \delta_1 = 0, \quad \varepsilon_2 = \gamma\alpha, \quad \delta_2 = (1-\gamma)/\beta$$

The closed-loop map from u to y is finite-gain $\mathcal{L}_2$ stable

The Small-Gain Theorem



$$\|y_{1\tau}\|_{\mathcal{L}} \leq \gamma_1 \|e_{1\tau}\|_{\mathcal{L}} + \beta_1, \quad \forall\ e_1 \in \mathcal{L}_e^m,\ \forall\ \tau \in [0, \infty)$$

$$\|y_{2\tau}\|_{\mathcal{L}} \leq \gamma_2 \|e_{2\tau}\|_{\mathcal{L}} + \beta_2, \quad \forall\ e_2 \in \mathcal{L}_e^q,\ \forall\ \tau \in [0, \infty)$$

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

Theorem 7.7

The feedback connection is finite-gain $\mathcal{L}$ stable if $\gamma_1\gamma_2 < 1$

Proof

$$e_{1\tau} = u_{1\tau} - (H_2 e_2)_\tau, \qquad e_{2\tau} = u_{2\tau} + (H_1 e_1)_\tau$$

$$\begin{aligned} \|e_{1\tau}\|_\mathcal{L} &\leq \|u_{1\tau}\|_\mathcal{L} + \|(H_2 e_2)_\tau\|_\mathcal{L} \\ &\leq \|u_{1\tau}\|_\mathcal{L} + \gamma_2 \|e_{2\tau}\|_\mathcal{L} + \beta_2 \end{aligned}$$

$$
\begin{aligned}
\|e_{1\tau}\|_{\mathcal{L}} &\leq \|u_{1\tau}\|_{\mathcal{L}} + \gamma_2\left(\|u_{2\tau}\|_{\mathcal{L}} + \gamma_1\|e_{1\tau}\|_{\mathcal{L}} + \beta_1\right) + \beta_2 \\
&= \gamma_1\gamma_2\|e_{1\tau}\|_{\mathcal{L}} \\
&\quad + \left(\|u_{1\tau}\|_{\mathcal{L}} + \gamma_2\|u_{2\tau}\|_{\mathcal{L}} + \beta_2 + \gamma_2\beta_1\right)
\end{aligned}
$$

$$
\|e_{1\tau}\|_{\mathcal{L}} \leq \frac{1}{1-\gamma_1\gamma_2}\left(\|u_{1\tau}\|_{\mathcal{L}} + \gamma_2\|u_{2\tau}\|_{\mathcal{L}} + \beta_2 + \gamma_2\beta_1\right)
$$

$$
\|e_{2\tau}\|_{\mathcal{L}} \leq \frac{1}{1-\gamma_1\gamma_2}\left(\|u_{2\tau}\|_{\mathcal{L}} + \gamma_1\|u_{1\tau}\|_{\mathcal{L}} + \beta_1 + \gamma_1\beta_2\right)
$$

$$
\|e_{\tau}\|_{\mathcal{L}} \leq \|e_{1\tau}\|_{\mathcal{L}} + \|e_{2\tau}\|_{\mathcal{L}}
$$

Example 7.9

$$H1: \quad \text{Hurwitz } G(s)$$

$$H2: \quad y_2 = \psi(t, e_2), \quad \|\psi(t, y)\| \le \gamma_2 \|y\|, \ \ \forall\ t, \forall\ y$$

H_1 is finite-gain $\mathcal{L}_2$ stable; $\mathcal{L}_2$ gain $\le \sup_{w \in R} \|G(j\omega)\|$

H_2 is finite-gain $\mathcal{L}_2$ stable; $\mathcal{L}_2$ gain $\le \gamma_2$

The feedback connection is finite-gain $\mathcal{L}_2$ stable if

$$\gamma_2 \sup_{w \in R} \|G(j\omega)\| < 1$$

Example 7.10

$$\dot{x} = f(t, x, v + d_1(t)), \quad \varepsilon \dot{z} = Az + B[u + d_2(t)], \quad v = Cz$$

A is a Hurwitz, $\quad -CA^{-1}B = I, \quad \varepsilon \ll 1, \quad d_i \in \mathcal{L}$

Model order reduction: $\varepsilon = 0$

$$v = -CA^{-1}B(u + d_2) = u + d_2$$

$$\dot{x} = f(t, x, u + d), \quad d = d_1 + d_2$$

Disturbance attenuation: Design $u = \phi(t, x)$ such that

$$\|x\|_{\mathcal{L}} \le \gamma \|d\|_{\mathcal{L}} + \beta, \quad \gamma < \delta$$

What happens when u is applied to the actual system?

$$\dot{x} = f(t, x, Cz + d_1(t)), \qquad \varepsilon \dot{z} = Az + B[\phi(t, x) + d_2(t)]$$

Assume $\dot{d}_2 \in \mathcal{L}$

$$\eta = z + A^{-1}B[\phi(t, x) + d_2(t)]$$

$$\dot{x} = f(t, x, \phi(t, x) + d(t) + C\eta), \qquad \varepsilon \dot{\eta} = A\eta + \varepsilon A^{-1}B[\dot{\phi} + \dot{d}_2(t)]$$

$$\dot{\phi} = \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x} f(t, x, \phi(t, x) + d(t) + C\eta)$$

The system can be represented as a feedback connection

$$H1: \quad \dot{x} = f(t, x, \phi(t, x) + e_1), \qquad y_1 = \dot{\phi}$$

$$H2: \quad \dot{\eta} = \frac{1}{\varepsilon} A\eta + A^{-1} B e_2, \qquad y_2 = -C\eta$$

$$u_1 = d_1 + d_2 = d, \quad u_2 = \dot{d}_2$$

Assume $\left\| \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x} f(t, x, \phi(t, x) + e_1) \right\| \le c_1 \|x\| + c_2 \|e_1\|$

$$\|y_1\|_{\mathcal{L}} \le \gamma_1 \|e_1\|_{\mathcal{L}} + \beta_1, \quad \gamma_1 = c_1\gamma + c_2, \quad \beta_1 = c_1 \beta$$

$$\|y_2\|_{\mathcal{L}} \le \varepsilon \gamma_f \|e_2\|_{\mathcal{L}} + \beta_2$$

By the small-gain theorem,

$$\|e_1\|_{\mathcal{L}} \le \frac{1}{1 - \varepsilon\gamma_1\gamma_f}[\|u_1\|_{\mathcal{L}} + \varepsilon\gamma_f\|u_2\|_{\mathcal{L}} + \varepsilon\gamma_f\beta_1 + \beta_2]$$

By design

$$\|x\|_{\mathcal{L}} \le \gamma\|e_1\|_{\mathcal{L}} + \beta$$

Hence

$$\|x\|_{\mathcal{L}} \le \underbrace{\frac{\gamma}{1 - \varepsilon\gamma_1\gamma_f}[\|d\|_{\mathcal{L}} + \varepsilon\gamma_f\|\dot{d}_2\|_{\mathcal{L}} + \varepsilon\gamma_f\beta_1 + \beta_2] + \beta}_{\to\gamma\|d\|_{\mathcal{L}}+\beta+\gamma\beta_2 \text{ as } \varepsilon\to 0}$$