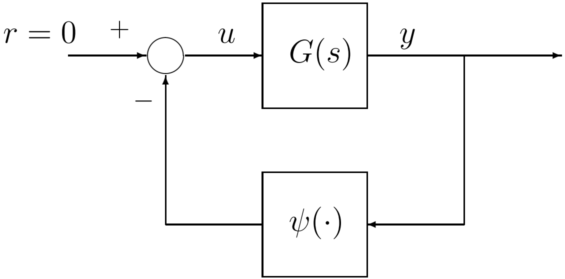
Nonlinear Control

Lecture # 18

Stability of Feedback Systems

Absolute Stability



Definition 7.1

The system is absolutely stable if the origin is globally uniformly asymptotically stable for any nonlinearity in a given sector. It is absolutely stable with finite domain if the origin is uniformly asymptotically stable

Circle Criterion

Suppose $G(s) = C(sI - A)^{-1}B + D$ is SPR, $\psi \in [0, \infty]$

$$\begin{aligned}
\dot{x} &= Ax + Bu \\
y &= Cx + Du \\
u &= -\psi(t, y)
\end{aligned}$$

By the KYP Lemma, $\exists\ P = P^T > 0,\ L, W, \varepsilon > 0$

$$\begin{aligned}
PA + A^T P &= -L^T L - \varepsilon P \\
PB &= C^T - L^T W \\
W^T W &= D + D^T
\end{aligned}$$

$$V(x) = \tfrac{1}{2} x^T P x$$

$$
\begin{aligned}
\dot{V} &= \tfrac{1}{2}x^T P\dot{x} + \tfrac{1}{2}\dot{x}^T P x \\
&= \tfrac{1}{2}x^T(PA + A^T P)x + x^T PBu \\
&= -\ \tfrac{1}{2}x^T L^T L x -\ \tfrac{1}{2}\varepsilon x^T P x + x^T(C^T - L^T W)u \\
&= -\ \tfrac{1}{2}x^T L^T L x -\ \tfrac{1}{2}\varepsilon x^T P x + (Cx + Du)^T u \\
&\quad - u^T D u - x^T L^T W u
\end{aligned}
$$

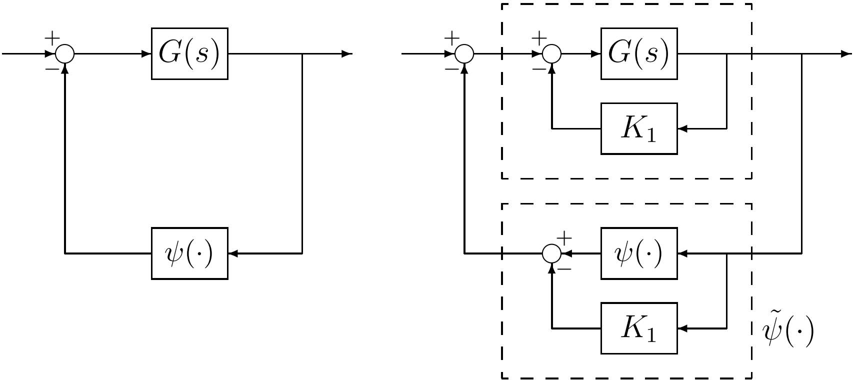
$$
u^T D u = \tfrac{1}{2}u^T(D + D^T)u = \tfrac{1}{2}u^T W^T W u
$$

$$
\dot{V} = -\ \tfrac{1}{2}\varepsilon x^T P x -\ \tfrac{1}{2}(Lx + Wu)^T(Lx + Wu) - y^T\psi(t, y)
$$

$$
y^T\psi(t, y) \ge 0 \quad \Rightarrow \quad \dot{V} \le -\ \tfrac{1}{2}\varepsilon x^T P x
$$

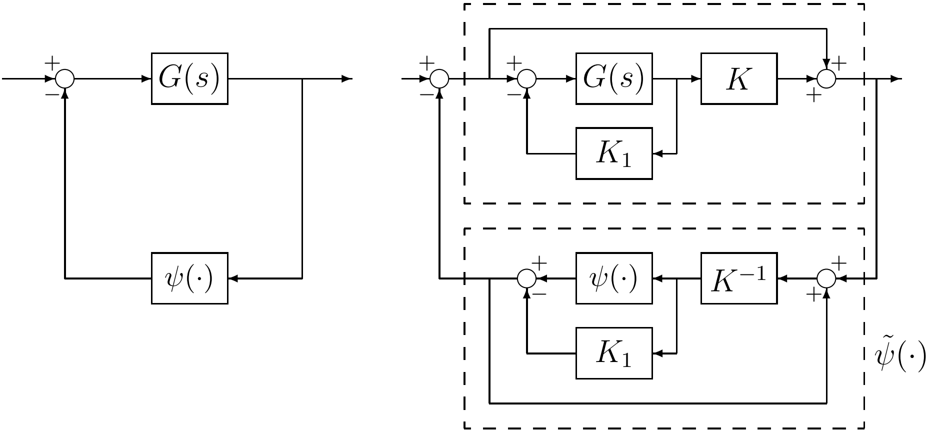
The origin is globally exponentially stable

What if $\psi \in [K_1, \infty]$?



$\tilde{\psi} \in [0, \infty]$; hence the origin is globally exponentially stable if $G(s)[I + K_1 G(s)]^{-1}$ is SPR

What if $\psi \in [K_1, K_2]$?



$\tilde{\psi} \in [0, \infty]$; hence the origin is globally exponentially stable if $I + KG(s)[I + K_1 G(s)]^{-1}$ is SPR

$$I + KG(s)[I + K_1G(s)]^{-1} = [I + K_2G(s)][I + K_1G(s)]^{-1}$$

Theorem 7.8 (Circle Criterion)

The system is absolutely stable if

- $\psi \in [K_1, \infty]$ and $G(s)[I + K_1G(s)]^{-1}$ is SPR, or
- $\psi \in [K_1, K_2]$ and $[I + K_2G(s)][I + K_1G(s)]^{-1}$ is SPR

If the sector condition is satisfied only on a set $Y \subset R^m$, then the foregoing conditions ensure absolute stability with finite domain

Example 7.11

$G(s)$ is Hurwitz, $G(\infty) = 0$

$$\|\psi(t, y)\| \leq \gamma_2 \|y\|, \quad \forall\ t,\ y$$

$$\psi \in [K_1, K_2], \quad K_1 = -\gamma_2 I, \quad K_2 = \gamma_2 I$$

By the circle criterion, the system is absolutely stable if

$$Z(s) = [I + \gamma_2 G(s)][I - \gamma_2 G(s)]^{-1} \text{ is SPR}$$

$$Z(\infty) + Z^T(\infty) = 2I$$

By Lemma 5.1,

$$Z(s) \text{ SPR } \Leftrightarrow\ Z(s) \text{ Hurwitz } \&\ Z(j\omega) + Z^T(-j\omega) > 0\ \forall\ \omega$$

$$Z(s) \text{ Hurwitz } \Leftrightarrow\ [I - \gamma_2 G(s)]^{-1} \text{ Hurwitz}$$

From the multivariable Nyquist criterion, $[I - \gamma_2 G(s)]^{-1}$ is Hurwitz if the plot of $\det[I - \gamma_2 G(j\omega)]$ does not go through nor encircle the origin. This will be the case if

$$\sigma_{\min}[I - \gamma_2 G(j\omega)] > 0$$

$$\sigma_{\min}[I - \gamma_2 G(j\omega)] \geq 1 - \gamma_1\gamma_2, \quad \gamma_1 = \sup_{\omega \in R} \|G(j\omega)\|$$

$$\gamma_1\gamma_2 < 1 \;\Rightarrow\; Z(s) \text{ Hurwitz}$$

$$Z(j\omega) + Z^T(-j\omega) = 2H^T(-j\omega)\left[I - \gamma_2^2 G^T(-j\omega)G(j\omega)\right]H(j\omega)$$

$$H(j\omega) = [I - \gamma_2 G(j\omega)]^{-1}$$

$$Z(j\omega) + Z^T(-j\omega) > 0 \;\Leftrightarrow\; \left[I - \gamma_2^2 G^T(-j\omega)G(j\omega)\right] > 0$$

$$\gamma_1\gamma_2 < 1 \;\Rightarrow\; Z(s) \text{ SPR}$$

Scalar Case: $\psi \in [\alpha, \beta],\ \beta > \alpha$

The system is absolutely stable if

$$\frac{1 + \beta G(s)}{1 + \alpha G(s)} \quad \text{is Hurwitz and}$$

$$\mathrm{Re}\left[\frac{1 + \beta G(j\omega)}{1 + \alpha G(j\omega)}\right] > 0, \quad \forall\ \omega \in [0, \infty]$$

Case 1: $\alpha > 0$

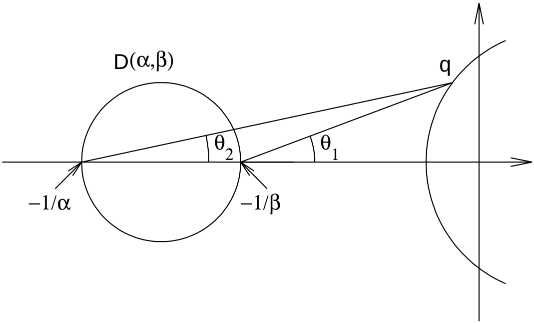
By the Nyquist criterion

$$\frac{1+\beta G(s)}{1+\alpha G(s)} = \frac{1}{1+\alpha G(s)} + \frac{\beta G(s)}{1+\alpha G(s)}$$

is Hurwitz if the Nyquist plot of $G(j\omega)$ does not intersect the point $-(1/\alpha) + j0$ and encircles it p times in the counterclockwise direction, where p is the number of poles of $G(s)$ in the open right-half complex plane

$$\frac{1+\beta G(j\omega)}{1+\alpha G(j\omega)} > 0 \quad \Leftrightarrow \quad \frac{\frac{1}{\beta} + G(j\omega)}{\frac{1}{\alpha} + G(j\omega)} > 0$$

$$\mathrm{Re}\left[\frac{\frac{1}{\beta}+G(j\omega)}{\frac{1}{\alpha}+G(j\omega)}\right] > 0, \quad \forall\ \omega \in [0,\infty]$$



The system is absolutely stable if the Nyquist plot of $G(j\omega)$ does not enter the disk $D(\alpha,\beta)$ and encircles it m times in the counterclockwise direction

Case 2: $\alpha = 0$

$$1 + \beta G(s)$$

$$\text{Re}[1 + \beta G(j\omega)] > 0, \quad \forall\, \omega \in [0, \infty]$$

$$\text{Re}[G(j\omega)] > -\frac{1}{\beta}, \quad \forall\, \omega \in [0, \infty]$$

The system is absolutely stable if $G(s)$ is Hurwitz and the Nyquist plot of $G(j\omega)$ lies to the right of the vertical line defined by $\text{Re}[s] = -1/\beta$

Case 3: $\alpha < 0 < \beta$

$$\mathrm{Re}\left[\frac{1+\beta G(j\omega)}{1+\alpha G(j\omega)}\right] > 0 \quad \Leftrightarrow \quad \mathrm{Re}\left[\frac{\frac{1}{\beta}+G(j\omega)}{\frac{1}{\alpha}+G(j\omega)}\right] < 0$$

The Nyquist plot of $G(j\omega)$ must lie inside the disk $D(\alpha, \beta)$.
The Nyquist plot cannot encircle the point $-(1/\alpha) + j0$.

From the Nyquist criterion, $G(s)$ must be Hurwitz

The system is absolutely stable if $G(s)$ is Hurwitz and the Nyquist plot of $G(j\omega)$ lies in the interior of the disk $D(\alpha, \beta)$

Theorem 7.9

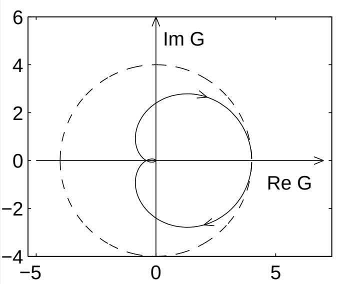
Consider an SISO $G(s)$ and $\psi \in [\alpha, \beta]$. Then, the system is absolutely stable if one of the following conditions is satisfied.

1. $0 < \alpha < \beta$, the Nyquist plot of $G(s)$ does not enter the disk $D(\alpha, \beta)$ and encircles it p times in the counterclockwise direction, where p is the number of poles of $G(s)$ with positive real parts
2. $0 = \alpha < \beta$, $G(s)$ is Hurwitz and the Nyquist plot of $G(s)$ lies to the right of the vertical line $\mathrm{Re}[s] = -1/\beta$.
3. $\alpha < 0 < \beta$, $G(s)$ is Hurwitz and the Nyquist plot of $G(s)$ lies in the interior of the disk $D(\alpha, \beta)$.

If the sector condition is satisfied only on an interval $[a, b]$, then the foregoing conditions ensure absolute stability with finite domain

Example 7.12

$$G(s) = \frac{24}{(s+1)(s+2)(s+3)}$$



Apply Case 3 with center $(0, 0)$ and radius $= 4$

Sector is $(-0.25, 0.25)$

Apply Case 3 with center $(1.5, 0)$ and radius $= 2.9$

Sector is $[-0.227, 0.714]$

Apply Case 2

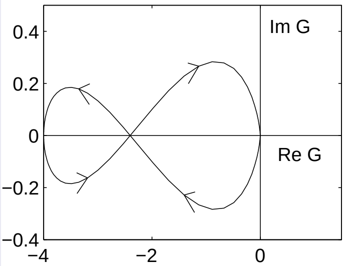
The Nyquist plot is to the right of $\text{Re}[s] = -0.857$

Sector is $[0, 1.166]$

$[0, 1.166]$ includes the saturation nonlinearity

Example 7.13

$$G(s) = \frac{24}{(s-1)(s+2)(s+3)}$$



G is not Hurwitz

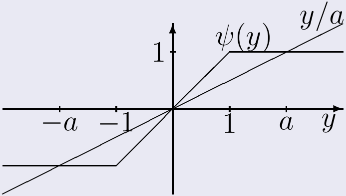
Apply Case 1

Center $= (-3.2, 0)$, Radius $= 0.1688 \Rightarrow \; [0.2969, 0.3298]$

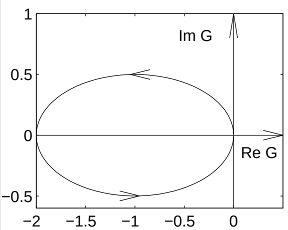
Example 7.14

$$G(s) = \frac{s+2}{(s+1)(s-1)}, \quad \psi(y) = \mathrm{sat}(y) \in [0, 1]$$

We cannot conclude absolute stability because case 1 requires $\alpha > 0$



$$-a \le y \le a \;\Rightarrow\; \psi \in [\alpha, 1], \quad \alpha = \frac{1}{a}$$



The Nyquist plot must encircle the disk $D(1/a, 1)$ once in the counterclockwise direction, which is satisfied for $a = 1.818$

The system is absolutely stable with finite domin

Estimate the region of attraction:

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = x_1 + u, \qquad y = 2x_1 + x_2$$

Loop transformation:

$$u = -\alpha y + \tilde{u}, \quad \tilde{y} = (\beta - \alpha)y + \tilde{u}, \quad \alpha = \frac{1}{a} = 0.55, \quad \beta = 1$$

$$\dot{x} = Ax + B\tilde{u}, \qquad \tilde{y} = Cx + D\tilde{u}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -0.1 & -0.55 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 0.9 & 0.45 \end{bmatrix}, D = 1$$

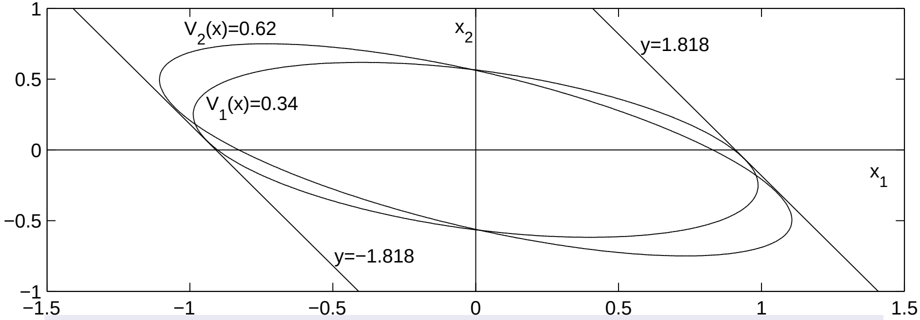
The KYP equations have two solutions

$$P_1 = \begin{bmatrix} 0.4946 & 0.4834 \\ 0.4834 & 1.0774 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 0.7595 & 0.4920 \\ 0.4920 & 1.9426 \end{bmatrix}$$

$$V_1(x) = x^T P_1 x, \quad V_2(x) = x^T P_2 x$$

$$\min_{\{|y|=1.818\}} V_1(x) = 0.3445, \qquad \min_{\{|y|=1.818\}} V_2(x) = 0.6212$$

$$\{V_1(x) \le 0.34\}, \quad \{V_2(x) \le 0.62\}$$



V2(x)=0.62
V1(x)=0.34
x2
x1
y=1.818
y=−1.818
1
0.5
0
−0.5
−1
−1.5
−1
−0.5
0
0.5
1
1.5