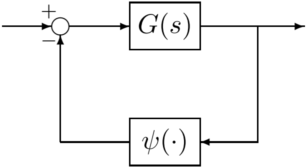
Nonlinear Control

Lecture # 19

Stability of Feedback Systems

Popov Criterion



$$\dot{x} = Ax + Bu, \quad y = Cx$$

(A, B) controllable, (A, C) observable

$$u_i = -\psi_i(y_i), \quad \psi_i \in [0, k_i], \quad 1 \leq i \leq m, \quad (0 < k_i \leq \infty)$$

$$G(s) = C(sI - A)^{-1}B$$

$$\Gamma = \text{diag}(\gamma_1, \ldots, \gamma_m), \quad M = \text{diag}(1/k_1, \cdots, 1/k_m)$$

Theorem 7.10

The system is absolutely stable if for $1 \le i \le m$,

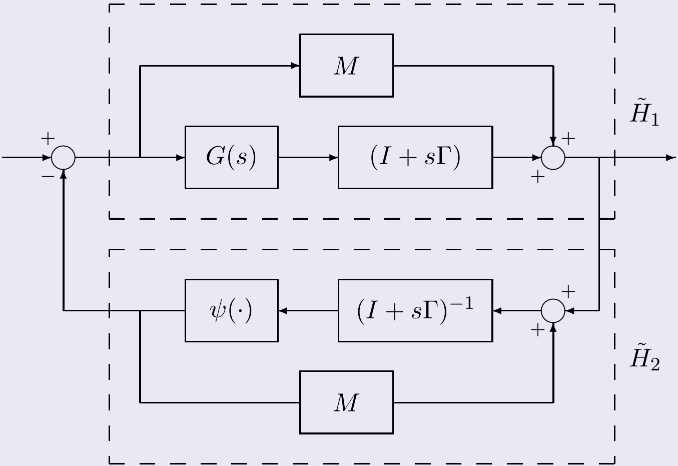
$$\psi_i \in [0, k_i], \quad 0 < k_i \le \infty$$

and there is $\gamma_i \ge 0$, with $(1 + \lambda_k \gamma_i) \neq 0$ for every eigenvalue λ_k of A, such that

$$M + (I + s\Gamma)G(s) \quad \text{is SPR}$$

If the sector condition $\psi_i \in [0, k_i]$ is satisfied only on a set $Y \subset R^m$, then the system is absolutely stable with finite domain

Proof



$\tilde{H}_1$:

$$\begin{aligned} & M + (I + s\Gamma)G(s) \\ & \quad = \;\; M + (I + s\Gamma)C(sI - A)^{-1}B \\ & \quad = \;\; M + C(sI - A)^{-1}B + \Gamma C s(sI - A)^{-1}B \\ & \quad = \;\; M + C(sI - A)^{-1}B + \Gamma C(sI - A + A)(sI - A)^{-1}B \\ & \quad = \;\; (C + \Gamma CA)(sI - A)^{-1}B + M + \Gamma CB \\ & \quad = \;\; \mathcal{C}(sI - \mathcal{A})^{-1}\mathcal{B} + \mathcal{D} \end{aligned}$$

$$\mathcal{A} = A, \quad \mathcal{B} = B, \quad \mathcal{C} = C + \Gamma CA, \quad \mathcal{D} = M + \Gamma CB$$

$$Av_k = \lambda_k v_k$$

$$(C + \Gamma CA)v_k = (C + \Gamma C\lambda_k)v_k = (I + \lambda_k\Gamma)Cv_k$$

$$(1+\lambda_k\gamma_i) \neq 0 \quad \Rightarrow \quad (\mathcal{A},\mathcal{C}) \text{ observable}$$

If $M + (I + s\Gamma)G(s)$ is SPR, by the KYP lemma, there are $P = P^T > 0$, L, and W, and $\varepsilon > 0$ that satisfy

$$\begin{aligned} PA + A^TP &= -L^TL - \varepsilon P \\ PB &= (C + \Gamma CA)^T - L^TW \\ W^TW &= 2M + \Gamma CB + B^TC^T\Gamma \end{aligned}$$

and $V_1 = \frac{1}{2}x^TPx$ is a storage function for $\tilde{H}_1$

$\tilde{H}_2$ consists of m decoupled components:

$$\gamma_i \dot{y}_i = -y_i + \frac{1}{k_i}\psi_i(y_i) + \tilde{e}_{2i}, \quad \tilde{y}_{2i} = \psi_i(y_i)$$

$$V_{2i} = \gamma_i \int_0^{y_i} \psi_i(\sigma)\ d\sigma$$

$$\begin{aligned} \dot{V}_{2i} &= \gamma_i \psi_i(y_i)\dot{y}_i \ = \ \psi_i(y_i)\left[-y_i + \tfrac{1}{k_i}\psi_i(y_i) + \tilde{e}_{2i}\right] \\ &= y_{2i}e_{2i} + \tfrac{1}{k_i}\psi_i(y_i)\left[\psi_i(y_i) - k_i y_i\right] \end{aligned}$$

$$\psi_i \in [0, k_i] \quad \Rightarrow \quad \psi_i(\psi_i - k_i y_i) \leq 0 \quad \Rightarrow \quad \dot{V}_{2i} \leq y_{2i}e_{2i}$$

$\tilde{H}_2$ is passive with the storage function

$$V_2 = \sum_{i=1}^{m} \gamma_i \int_0^{y_i} \psi_i(\sigma)\ d\sigma$$

Use $$V = \tfrac{1}{2}x^T P x + \sum_{i=1}^{m} \gamma_i \int_0^{y_i} \psi_i(\sigma)\ d\sigma$$

as a Lyapunov function candidate for the original feedback connection

$$\dot{x} = Ax + Bu, \quad y = Cx, \quad u = -\psi(y)$$

$$\begin{array}{rcl} \dot{V} & = & \tfrac{1}{2}x^T P\dot{x} + \tfrac{1}{2}\dot{x}^T P x + \psi^T(y)\Gamma\dot{y} \\ & = & \tfrac{1}{2}x^T(PA + A^T P)x + x^T PBu \\ & & + \psi^T(y)\Gamma C(Ax + Bu) \\ & = & -\ \tfrac{1}{2}x^T L^T Lx -\ \tfrac{1}{2}\varepsilon x^T P x \\ & & + x^T(C^T + A^T C^T\Gamma - L^T W)u \\ & & + \psi^T(y)\Gamma CAx + \psi^T(y)\Gamma CBu \end{array}$$

$$
\begin{aligned}
\dot{V} &= -\tfrac{1}{2}\varepsilon x^T P x - \tfrac{1}{2}(Lx + Wu)^T(Lx + Wu) \\
&\quad - \psi(y)^T[y - M\psi(y)] \\
&\leq -\tfrac{1}{2}\varepsilon x^T P x - \psi(y)^T[y - M\psi(y)]
\end{aligned}
$$

$$
\psi_i \in [0, k_i] \;\Rightarrow\; \psi(y)^T[y - M\psi(y)] \geq 0 \;\Rightarrow\; \dot{V} \leq -\tfrac{1}{2}\varepsilon x^T P x
$$

The origin is globally asymptotically stable

Scalar case

$$\frac{1}{k} + (1 + s\gamma)G(s)$$

is SPR if $G(s)$ is Hurwitz and

$$\frac{1}{k} + \mathrm{Re}[G(j\omega)] - \gamma\omega\mathrm{Im}[G(j\omega)] > 0, \quad \forall\ \omega \in [0, \infty)$$

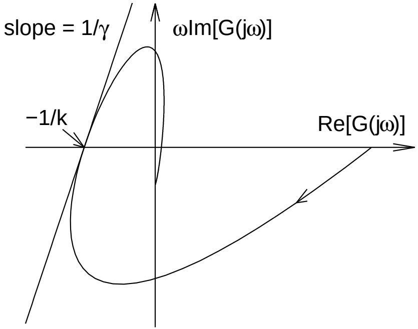
If

$$\lim_{\omega\to\infty} \left\{ \frac{1}{k} + \mathrm{Re}[G(j\omega)] - \gamma\omega\mathrm{Im}[G(j\omega)] \right\} = 0$$

we also need

$$\lim_{\omega\to\infty} \omega^2 \left\{ \frac{1}{k} + \mathrm{Re}[G(j\omega)] - \gamma\omega\mathrm{Im}[G(j\omega)] \right\} > 0$$

$$\frac{1}{k} + \mathrm{Re}[G(j\omega)] - \gamma\omega \mathrm{Im}[G(j\omega)] > 0, \quad \forall\ \omega \in [0, \infty)$$



Popov Plot

Example

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_2 - h(y), \quad y = x_1$$

$$\dot{x}_2 = -\alpha x_1 - x_2 - h(y) + \alpha x_1, \quad \alpha > 0$$

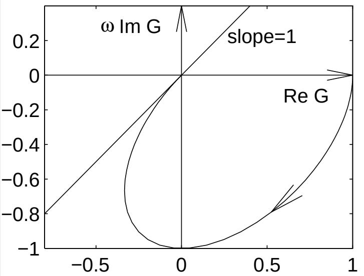
$$G(s) = \frac{1}{s^2 + s + \alpha}, \qquad \psi(y) = h(y) - \alpha y$$

$$h \in [\alpha, \beta] \;\; \Rightarrow \;\; \psi \in [0, k] \qquad\quad (k = \beta - \alpha > 0)$$

$$\gamma > 1 \;\; \Rightarrow \;\; \frac{\alpha - \omega^2 + \gamma\omega^2}{(\alpha - \omega^2)^2 + \omega^2} > 0, \;\; \forall\, \omega \in [0, \infty)$$

and $$\lim_{\omega \to \infty} \frac{\omega^2(\alpha - \omega^2 + \gamma\omega^2)}{(\alpha - \omega^2)^2 + \omega^2} = \gamma - 1 > 0$$

The system is absolutely stable for $\psi \in [0, \infty]$ $(h \in [\alpha, \infty])$



Compare with the circle criterion $(\gamma = 0)$

$$\frac{1}{k} + \frac{\alpha - \omega^2}{(\alpha - \omega^2)^2 + \omega^2} > 0, \quad \forall\, \omega \in [0, \infty], \quad \text{for } k < 1 + 2\sqrt{\alpha}$$