Nonlinear Control

Lecture # 2

Two-Dimensional Systems

$$
\begin{aligned}
\dot{x}_1 &= f_1(x_1, x_2) = f_1(x) \\
\dot{x}_2 &= f_2(x_1, x_2) = f_2(x)
\end{aligned}
$$

Let $x(t) = (x_1(t), x_2(t))$ be a solution that starts at initial state $x_0 = (x_{10}, x_{20})$. The locus in the x_1–x_2 plane of the solution $x(t)$ for all $t \geq 0$ is a curve that passes through the point x_0. This curve is called a *trajectory* or *orbit*

The x_1–x_2 plane is called the *state plane* or *phase plane*

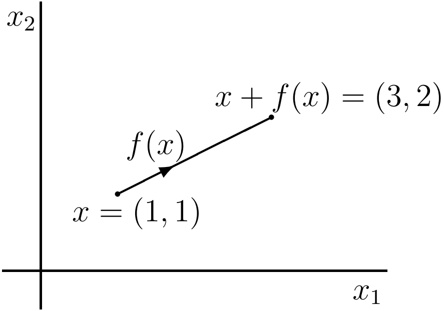
The family of all trajectories is called the *phase portrait*

The *vector field* $f(x) = (f_1(x), f_2(x))$ is tangent to the trajectory at point x because

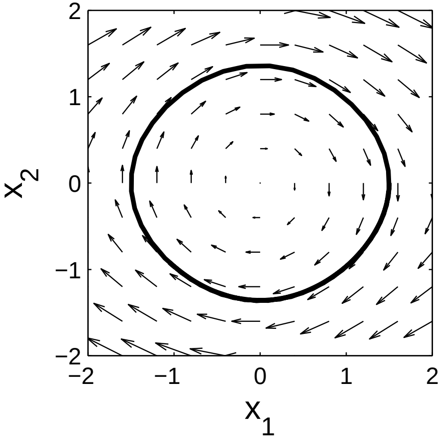
$$
\frac{dx_2}{dx_1} = \frac{f_2(x)}{f_1(x)}
$$

Vector Field diagram

Represent $f(x)$ as a vector based at x; that is, assign to x the directed line segment from x to $x + f(x)$



Repeat at every point in a grid covering the plane



$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\sin x_1$$

Numerical Construction of the Phase Portrait

- Select a bounding box in the state plane
- Select an initial point x_0 and calculate the trajectory through it by solving

$$\dot{x} = f(x), \quad x(0) = x_0$$

in forward time (with positive t) and in reverse time (with negative t)

$$\dot{x} = -f(x), \quad x(0) = x_0$$

- Repeat the process interactively
- Use Simulink or pplane

Qualitative Behavior of Linear Systems

$$\dot{x} = Ax, \qquad A \text{ is a } 2 \times 2 \text{ real matrix}$$

$$x(t) = M \exp(J_r t) M^{-1} x_0$$

When A has distinct eigenvalues,

$$J_r = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

$$x(t) = Mz(t)$$

$$\dot{z} = J_r z(t)$$

Case 1. Both eigenvalues are real:

$$M = [v_1, v_2]$$

v_1 & v_2 are the real eigenvectors associated with λ_1 & λ_2

$$\dot{z}_1 = \lambda_1 z_1, \qquad \dot{z}_2 = \lambda_2 z_2$$

$$z_1(t) = z_{10} e^{\lambda_1 t}, \qquad z_2(t) = z_{20} e^{\lambda_2 t}$$

$$z_2 = c z_1^{\lambda_2/\lambda_1}, \quad c = z_{20}/(z_{10})^{\lambda_2/\lambda_1}$$

The shape of the phase portrait depends on the signs of λ_1 and λ_2

$$\lambda_2 < \lambda_1 < 0$$

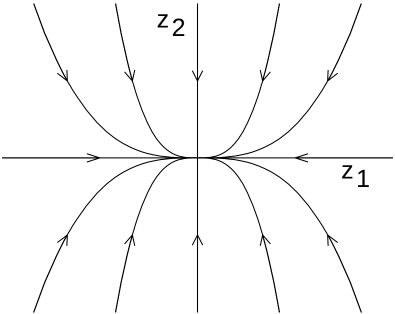
$e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ tend to zero as $t \to \infty$

$e^{\lambda_2 t}$ tends to zero faster than $e^{\lambda_1 t}$

Call λ_2 the fast eigenvalue (v_2 the fast eigenvector) and λ_1 the slow eigenvalue (v_1 the slow eigenvector)

The trajectory tends to the origin along the curve $z_2 = c z_1^{\lambda_2/\lambda_1}$ with $\lambda_2/\lambda_1 > 1$

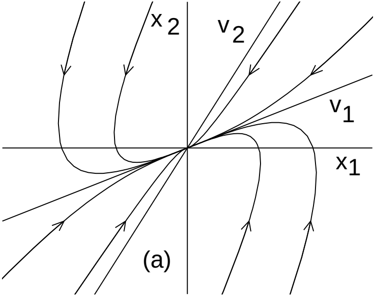
$$\frac{dz_2}{dz_1} = c\frac{\lambda_2}{\lambda_1} z_1^{[(\lambda_2/\lambda_1)-1]}$$



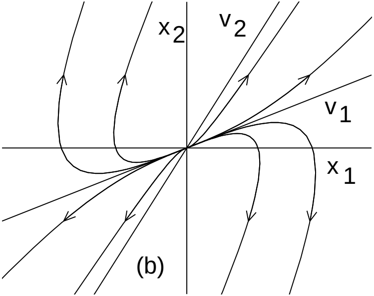
Stable Node

$$\lambda_2 > \lambda_1 > 0$$

Reverse arrowheads $\Rightarrow$ Unstable Node



Stable Node



Unstable Node

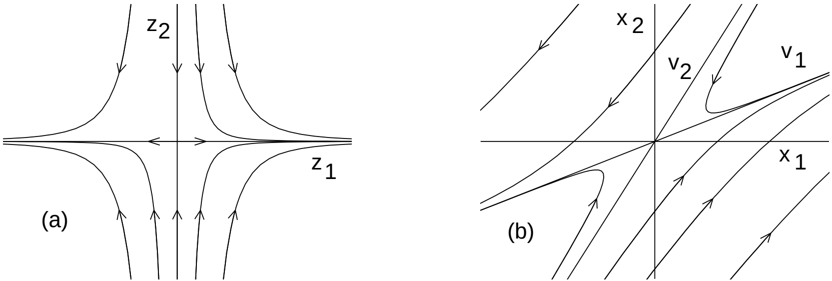
$$\lambda_2 < 0 < \lambda_1$$

$e^{\lambda_1 t} \to \infty$, while $e^{\lambda_2 t} \to 0$ as $t \to \infty$

Call λ_2 the stable eigenvalue (v_2 the stable eigenvector) and λ_1 the unstable eigenvalue (v_1 the unstable eigenvector)

$$z_2 = c z_1^{\lambda_2/\lambda_1}, \qquad \lambda_2/\lambda_1 < 0$$

Saddle



Phase Portrait of a Saddle Point

Case 2. Complex eigenvalues: $\lambda_{1,2} = \alpha \pm j\beta$

$$\dot{z}_1 = \alpha z_1 - \beta z_2, \qquad \dot{z}_2 = \beta z_1 + \alpha z_2$$

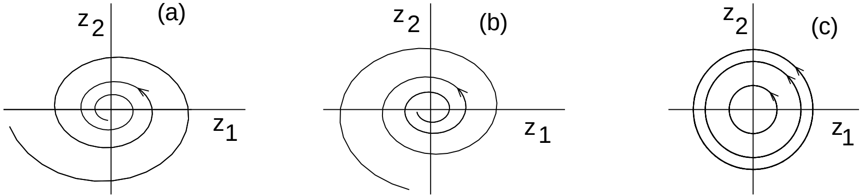
$$r = \sqrt{z_1^2 + z_2^2}, \qquad \theta = \tan^{-1}\left(\frac{z_2}{z_1}\right)$$

$$r(t) = r_0 e^{\alpha t} \quad \text{and} \quad \theta(t) = \theta_0 + \beta t$$

$$\alpha < 0 \quad \Rightarrow \quad r(t) \to 0 \text{ as } t \to \infty$$

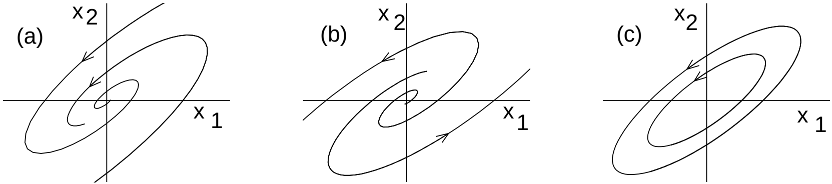
$$\alpha > 0 \quad \Rightarrow \quad r(t) \to \infty \text{ as } t \to \infty$$

$$\alpha = 0 \quad \Rightarrow \quad r(t) \equiv r_0 \ \forall \ t$$



$\alpha < 0$ $\alpha > 0$ $\alpha = 0$

Stable Focus Unstable Focus Center



Effect of Perturbations

$$A \to A + \delta A \qquad (\delta A \text{ arbitrarily small})$$

The eigenvalues of a matrix depend continuously on its parameters

A node (with distinct eigenvalues), a saddle or a focus is structurally stable because the qualitative behavior remains the same under arbitrarily small perturbations in A

A center is not structurally stable

$$\begin{bmatrix} \mu & 1 \\ -1 & \mu \end{bmatrix}, \qquad \text{Eigenvalues} = \mu \pm j$$

$$\mu < 0 \quad \Rightarrow \quad \text{Stable Focus}, \qquad \mu > 0 \quad \Rightarrow \quad \text{Unstable Focus}$$

Qualitative Behavior Near Equilibrium Points

The qualitative behavior of a nonlinear system near an equilibrium point can take one of the patterns we have seen with linear systems. Correspondingly the equilibrium points are classified as stable node, unstable node, saddle, stable focus, unstable focus, or center

Can we determine the type of the equilibrium point of a nonlinear system by linearization?

Let $p = (p_1, p_2)$ be an equilibrium point of the system

$$\dot{x}_1 = f_1(x_1, x_2), \qquad \dot{x}_2 = f_2(x_1, x_2)$$

where f_1 and f_2 are continuously differentiable
Expand f_1 and f_2 in Taylor series about (p_1, p_2)

$$\begin{aligned}
\dot{x}_1 &= f_1(p_1, p_2) + a_{11}(x_1 - p_1) + a_{12}(x_2 - p_2) + \text{H.O.T.} \\
\dot{x}_2 &= f_2(p_1, p_2) + a_{21}(x_1 - p_1) + a_{22}(x_2 - p_2) + \text{H.O.T.}
\end{aligned}$$

$$a_{11} = \left.\frac{\partial f_1(x_1, x_2)}{\partial x_1}\right|_{x=p}, \qquad a_{12} = \left.\frac{\partial f_1(x_1, x_2)}{\partial x_2}\right|_{x=p}$$

$$a_{21} = \left.\frac{\partial f_2(x_1, x_2)}{\partial x_1}\right|_{x=p}, \qquad a_{22} = \left.\frac{\partial f_2(x_1, x_2)}{\partial x_2}\right|_{x=p}$$

$$f_1(p_1, p_2) = f_2(p_1, p_2) = 0$$

$$y_1 = x_1 - p_1 \qquad y_2 = x_2 - p_2$$

$$\begin{aligned}
\dot{y}_1 &= \dot{x}_1 = a_{11} y_1 + a_{12} y_2 + \text{H.O.T.} \\
\dot{y}_2 &= \dot{x}_2 = a_{21} y_1 + a_{22} y_2 + \text{H.O.T.}
\end{aligned}$$

$$\dot{y} \approx A y$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ & \\ a_{21} & a_{22} \end{bmatrix} = \left. \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ & \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \right|_{x=p} = \left. \frac{\partial f}{\partial x} \right|_{x=p}$$

Eigenvalues of A	Type of equilibrium point of the nonlinear system
$\lambda_2 < \lambda_1 < 0$	Stable Node
$\lambda_2 > \lambda_1 > 0$	Unstable Node
$\lambda_2 < 0 < \lambda_1$	Saddle
$\alpha \pm j\beta,\ \alpha < 0$	Stable Focus
$\alpha \pm j\beta,\ \alpha > 0$	Unstable Focus
$\pm j\beta$	Linearization Fails

Example 2.1

$$\begin{aligned}\dot{x}_1 &= -x_2 - \mu x_1(x_1^2 + x_2^2) \\ \dot{x}_2 &= x_1 - \mu x_2(x_1^2 + x_2^2)\end{aligned}$$

$x = 0$ is an equilibrium point

$$\frac{\partial f}{\partial x} = \begin{bmatrix} -\mu(3x_1^2 + x_2^2) & -(1 + 2\mu x_1 x_2) \\ (1 - 2\mu x_1 x_2) & -\mu(x_1^2 + 3x_2^2) \end{bmatrix}$$

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$x_1 = r\cos\theta \text{ and } x_2 = r\sin\theta \;\Rightarrow\; \dot{r} = -\mu r^3 \text{ and } \dot{\theta} = 1$$

Stable focus when $\mu > 0$ and Unstable focus when $\mu < 0$

For a saddle point, we can use linearization to generate the stable and unstable trajectories

Let the eigenvalues of the linearization be $\lambda_1 > 0 > \lambda_2$ and the corresponding eigenvectors be v_1 and v_2

The stable and unstable trajectories will be tangent to the stable and unstable eigenvectors, respectively, as they approach the equilibrium point p

For the unstable trajectories use $x_0 = p \pm \alpha v_1$

For the stable trajectories use $x_0 = p \pm \alpha v_2$

α is a small positive number