Nonlinear Control

Lecture # 20

Special nonlinear Forms

Normal Form

Relative Degree

$$\dot{x} = f(x) + g(x)u, \qquad y = h(x)$$

where f, g, and h are sufficiently smooth in a domain D
$f : D \to R^n$ and $g : D \to R^n$ are called vector fields on D

$$\dot{y} = \frac{\partial h}{\partial x}[f(x) + g(x)u] \stackrel{\text{def}}{=} L_f h(x) + L_g h(x)\ u$$

$$L_f h(x) = \frac{\partial h}{\partial x} f(x)$$

is the *Lie Derivative* of h with respect to f or along f

$$L_g L_f h(x) = \frac{\partial (L_f h)}{\partial x} g(x)$$

$$L_f^2 h(x) = L_f L_f h(x) = \frac{\partial (L_f h)}{\partial x} f(x)$$

$$L_f^k h(x) = L_f L_f^{k-1} h(x) = \frac{\partial (L_f^{k-1} h)}{\partial x} f(x)$$

$$L_f^0 h(x) = h(x)$$

$$\dot{y} = L_f h(x) + L_g h(x)\ u$$

$$L_g h(x) = 0 \quad \Rightarrow \quad \dot{y} = L_f h(x)$$

$$y^{(2)} = \frac{\partial (L_f h)}{\partial x} [f(x) + g(x)u] = L_f^2 h(x) + L_g L_f h(x)\ u$$

$$L_g L_f h(x) = 0 \quad \Rightarrow \quad y^{(2)} = L_f^2 h(x)$$

$$y^{(3)} = L_f^3 h(x) + L_g L_f^2 h(x)\ u$$

$$L_g L_f^{i-1} h(x) = 0, \quad i = 1, 2, \ldots, \rho - 1; \qquad L_g L_f^{\rho-1} h(x) \neq 0$$

$$y^{(\rho)} = L_f^{\rho} h(x) + L_g L_f^{\rho-1} h(x)\ u$$

Definition 8.1

The system

$$\dot{x} = f(x) + g(x)u, \qquad y = h(x)$$

has relative degree ρ, $1 \leq \rho \leq n$, in $\mathcal{R} \subset D$ if $\forall\ x \in \mathcal{R}$

$$L_g L_f^{i-1} h(x) = 0, \quad i = 1, 2, \ldots, \rho - 1; \qquad L_g L_f^{\rho-1} h(x) \neq 0$$

Example 8.1

Controlled van der Pol equation

$$\dot{x}_1 = x_2/\varepsilon, \quad \dot{x}_2 = \varepsilon[-x_1 + x_2 - \tfrac{1}{3}x_2^3 + u], \quad y = x_1$$

$$\dot{y} = \dot{x}_1 = x_2/\varepsilon, \quad \ddot{y} = \dot{x}_2/\varepsilon = -x_1 + x_2 - \tfrac{1}{3}x_2^3 + u$$

Relative degree two over R^2

$$\dot{x}_1 = x_2/\varepsilon, \quad \dot{x}_2 = \varepsilon[-x_1 + x_2 - \tfrac{1}{3}x_2^3 + u], \quad y = x_2$$

$\dot{y} = \varepsilon[-x_1 + x_2 - \tfrac{1}{3}x_2^3 + u]$, Relative degree one over R^2

$$\dot{x}_1 = x_2/\varepsilon, \quad \dot{x}_2 = \varepsilon[-x_1 + x_2 - \tfrac{1}{3}x_2^3 + u], \quad y = \tfrac{1}{2}(\varepsilon^2 x_1^2 + x_2^2)$$

$$\dot{y} = \varepsilon^2 x_1 \dot{x}_1 + x_2 \dot{x}_2 = \varepsilon x_2^2 - (\varepsilon/3)x_2^4 + \varepsilon x_2 u$$

Relative degree one in $\{x_2 \neq 0\}$

Example 8.2 (Field-controlled DC motor)

$$\begin{aligned}
\dot{x}_1 &= d_1(-x_1 - x_2 x_3 + V_a) \\
\dot{x}_2 &= d_2[-f_e(x_2) + u] \\
\dot{x}_3 &= d_3(x_1 x_2 - b x_3) \\
y &= x_3
\end{aligned}$$

$$\dot{y} = \dot{x}_3 = d_3(x_1 x_2 - b x_3)$$

$$\ddot{y} = d_3(x_1 \dot{x}_2 + \dot{x}_1 x_2 - b\dot{x}_3) = (\cdots) + d_2 d_3 x_1 u$$

Relative degree two in $\{x_1 \neq 0\}$

Example 8.3

$$H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_0}$$

$$\dot{x} = Ax + Bu, \qquad y = Cx$$

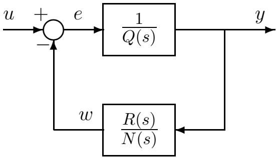
$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & & & \dots & 0 \\ 0 & 0 & 1 & \dots & & & \dots & 0 \\ \vdots & & \ddots & & & & & \vdots \\ & & & \ddots & & & & \\ & & & & \ddots & & & \vdots \\ \vdots & & & & & \ddots & & 0 \\ 0 & & & & & & 0 & 1 \\ -a_0 & -a_1 & \dots & \dots & -a_m & \dots & \dots & -a_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} b_0 & b_1 & \dots & \dots & b_m & 0 & \dots & 0 \end{bmatrix}$$

$$\dot{y} = CAx + CBu, \quad \text{If } m = n-1,\ CB = b_{n-1} \neq 0 \ \Rightarrow\ \rho = 1$$

$$CA^{i-1}B = 0,\ i = 1, \ldots, n-m-1,\ CA^{n-m-1}B = b_m \neq 0$$
$$y^{(n-m)} = CA^{n-m}x + CA^{n-m-1}Bu \ \Rightarrow\ \rho = n-m$$

$$H(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{Q(s)N(s) + R(s)} = \frac{\frac{1}{Q(s)}}{1 + \frac{1}{Q(s)}\ \frac{R(s)}{N(s)}}$$



State model of $1/Q(s)$: $\xi = \mathrm{col}\left(y, \dot{y}, \ldots, y^{(\rho-1)}\right)$

$$\dot{\xi} = (A_c + B_c\lambda^T)\xi + B_c b_m e, \qquad y = C_c\xi$$

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & 0 & 1 \\ 0 & \ldots & \ldots & 0 & 0 \end{bmatrix}, \; B_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \; C_c = \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 \end{bmatrix}$$

State model of $R(s)/N(s)$

$$\dot{\eta} = A_0\eta + B_0 y, \qquad w = C_0\eta$$

State model of $H(s)$

$$\begin{aligned}
\dot{\eta} &= A_0\eta + B_0 C_c \xi \\
\dot{\xi} &= A_c\xi + B_c(\lambda^T \xi - b_m C_0 \eta + b_m u) \\
y &= C_c \xi
\end{aligned}$$

The eigenvalues of A_0 are the zeros of $H(s)$

Change of variables:

$$z = T(x) = \begin{bmatrix} \phi_1(x) \\ \vdots \\ \phi_{n-\rho}(x) \\ --- \\ h(x) \\ \vdots \\ L_f^{\rho-1}h(x) \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} \phi(x) \\ --- \\ \psi(x) \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} \eta \\ --- \\ \xi \end{bmatrix}$$

ϕ_1 to $\phi_{n-\rho}$ are chosen such that $T(x)$ is a diffeomorphism on a domain $D_x \subset \mathcal{R}$

$$\text{When } \rho = n, \quad z = T(x) = \psi(x) = \xi$$

$$
\begin{aligned}
\dot{\eta} &= \frac{\partial \phi}{\partial x}[f(x) + g(x)u] \;=\; f_0(\eta, \xi) + g_0(\eta, \xi)u \\
\dot{\xi}_i &= \xi_{i+1}, \quad 1 \le i \le \rho - 1 \\
\dot{\xi}_\rho &= L_f^\rho h(x) + L_g L_f^{\rho-1} h(x)\; u \\
y &= \xi_1
\end{aligned}
$$

Choose $\phi(x)$ such that $T(x)$ is a diffeomorphism and

$$
\frac{\partial \phi_i}{\partial x} g(x) = 0, \quad \text{for } 1 \le i \le n - \rho,\ \forall\ x \in D_x
$$

Always possible (at least locally)

$$
\dot{\eta} = f_0(\eta, \xi)
$$

Theorem 8.1

Suppose the system

$$\dot{x} = f(x) + g(x)u, \qquad y = h(x)$$

has relative degree ρ ($\leq n$) in $\mathcal{R}$. If $\rho = n$, then for every $x_0 \in \mathcal{R}$, a neighborhood N of x_0 exists such that the map $T(x) = \psi(x)$, restricted to N, is a diffeomorphism on N. If $\rho < n$, then, for every $x_0 \in \mathcal{R}$, a neighborhood N of x_0 and smooth functions $\phi_1(x), \ldots, \phi_{n-\rho}(x)$ exist such that

$$\frac{\partial \phi_i}{\partial x} g(x) = 0, \quad \text{for } 1 \leq i \leq n - \rho$$

is satisfied for all $x \in N$ and the map $T(x) = \left[\begin{array}{c} \phi(x) \\ \psi(x) \end{array} \right]$, restricted to N, is a diffeomorphism on N

Normal Form:

$$\begin{aligned} \dot{\eta} &= f_0(\eta, \xi) \\ \dot{\xi}_i &= \xi_{i+1}, \quad 1 \le i \le \rho - 1 \\ \dot{\xi}_\rho &= L_f^\rho h(x) + L_g L_f^{\rho-1} h(x)\ u \\ y &= \xi_1 \end{aligned}$$

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & 0 & 1 \\ 0 & \dots & \dots & 0 & 0 \end{bmatrix}, \quad B_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C_c = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \end{bmatrix}$$

$$
\begin{aligned}
\dot{\eta} &= f_0(\eta, \xi) \\
\dot{\xi} &= A_c\xi + B_c\left[L_f^{\rho}h(x) + L_gL_f^{\rho-1}h(x)\; u\right] \\
y &= C_c\xi
\end{aligned}
$$

$$
\tilde{\psi}(\eta, \xi) = \left. L_f^{\rho}h(x)\right|_{x=T^{-1}(z)}, \tilde{\gamma}(\eta, \xi) = \left. L_gL_f^{\rho-1}h(x)\right|_{x=T^{-1}(z)}
$$

$$
\dot{\xi} = A_c\xi + B_c[\tilde{\psi}(\eta, \xi) + \tilde{\gamma}(\eta, \xi)u]
$$

If x^* is an open-loop equilibrium point at which $y = 0$; i.e., $f(x^*) = 0$ and $h(x^*) = 0$, then $\psi(x^*) = 0$. Take $\phi(x^*) = 0$ so that $z = 0$ is an open-loop equilibrium point.

Zero Dynamics

$$\begin{aligned} \dot{\eta} &= f_0(\eta, \xi) \\ \dot{\xi} &= A_c\xi + B_c\left[L_f^\rho h(x) + L_g L_f^{\rho-1} h(x)\ u\right] \\ y &= C_c\xi \end{aligned}$$

$$y(t) \equiv 0 \ \Rightarrow \ \xi(t) \equiv 0 \ \Rightarrow \ u(t) \equiv -\,\frac{L_f^\rho h(x(t))}{L_g L_f^{\rho-1} h(x(t))}$$

$$\Rightarrow \ \dot{\eta} = f_0(\eta, 0)$$

Definition

The equation $\dot{\eta} = f_0(\eta, 0)$ is called the *zero dynamics* of the system. The system is said to be *minimum phase* if the zero dynamics have an asymptotically stable equilibrium point in the domain of interest (at the origin if $T(0) = 0$)

$$Z^* = \{x \in \mathcal{R} \mid h(x) = L_f h(x) = \cdots = L_f^{\rho-1} h(x) = 0\}$$

$$y(t) \equiv 0 \;\Rightarrow\; x(t) \in Z^*$$

$$\Rightarrow\; u = u^*(x) \stackrel{\text{def}}{=} -\left.\frac{L_f^\rho h(x)}{L_g L_f^{\rho-1} h(x)}\right|_{x \in Z^*}$$

The restricted motion of the system is described by

$$\dot{x} = f^*(x) \stackrel{\text{def}}{=} \left[f(x) - g(x)\frac{L_f^\rho h(x)}{L_g L_f^{\rho-1} h(x)}\right]_{x \in Z^*}$$

Example 8.4

$$\dot{x}_1 = x_2/\varepsilon, \quad \dot{x}_2 = \varepsilon[-x_1 + x_2 - \tfrac{1}{3}x_2^3 + u], \quad y = x_2$$

$$\dot{y} = \dot{x}_2 = \varepsilon[-x_1 + x_2 - \tfrac{1}{3}x_2^3 + u] \;\Rightarrow\; \rho = 1$$

The system is in the normal form with $\eta = x_1$ and $\xi = x_2$

$$y(t) \equiv 0 \;\Rightarrow x_2(t) \equiv 0 \;\Rightarrow\; \dot{x}_1 = 0$$

Non-minimum phase

Example 8.5

$$\dot{x}_1 = -x_1 + \frac{2 + x_3^2}{1 + x_3^2}\, u, \;\; \dot{x}_2 = x_3, \;\; \dot{x}_3 = x_1 x_3 + u, \;\; y = x_2$$

$$\dot{y} = \dot{x}_2 = x_3$$

$$\ddot{y} = \dot{x}_3 = x_1 x_3 + u \quad \Rightarrow \quad \rho = 2$$

$$Z^* = \{x_2 = x_3 = 0\}$$

$$u = u^*(x) = 0 \quad \Rightarrow \quad \dot{x}_1 = -x_1$$

Minimum phase

Find $\phi(x)$ such that

$$\phi(0) = 0, \quad \frac{\partial \phi}{\partial x} g(x) = \left[\begin{array}{ccc} \frac{\partial \phi}{\partial x_1}, & \frac{\partial \phi}{\partial x_2}, & \frac{\partial \phi}{\partial x_3} \end{array} \right] \left[\begin{array}{c} \frac{2+x_3^2}{1+x_3^2} \\ 0 \\ 1 \end{array} \right] = 0$$

and

$$T(x) = \left[\begin{array}{ccc} \phi(x) & x_2 & x_3 \end{array} \right]^T$$

is a diffeomorphism

$$\frac{\partial \phi}{\partial x_1} \cdot \frac{2 + x_3^2}{1 + x_3^2} + \frac{\partial \phi}{\partial x_3} = 0$$

$$\phi(x) = x_1 - x_3 - \tan^{-1} x_3$$

$$T(x) = \begin{bmatrix} x_1 - x_3 - \tan^{-1} x_3 \\ x_2 \\ x_3 \end{bmatrix}, \quad \frac{\partial T}{\partial x} = \begin{bmatrix} 1 & 0 & \star \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$T(x)$ is a global diffeomorphism

$$\begin{aligned}
\dot{\eta} &= -\left(\eta + \xi_2 + \tan^{-1} \xi_2\right) \left(1 + \frac{2 + \xi_2^2}{1 + \xi_2^2} \xi_2\right) \\
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \left(\eta + \xi_2 + \tan^{-1} \xi_2\right) \xi_2 + u \\
y &= \xi_1
\end{aligned}$$