Nonlinear Control

Lecture # 22

Special nonlinear Forms

Observer Form

Definition

A nonlinear system is in the observer form if

$$\dot{x} = Ax + \psi(y, u), \qquad y = Cx$$

where (A, C) is observable

Observer:

$$\dot{\hat{x}} = A\hat{x} + \psi(y, u) + H(y - C\hat{x})$$

$$\tilde{x} = x - \hat{x}$$

$$\dot{\tilde{x}} = (A - HC)\tilde{x}$$

Design H such that $(A - HC)$ is Hurwitz

Example 8.15 (A single link manipulator with flexible joints)

$$\dot{x} = \begin{bmatrix} x_2 \\ -a \sin x_1 - b(x_1 - x_3) \\ x_4 \\ c(x_1 - x_3) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ d \end{bmatrix} u, \quad y = x_1$$

$$\dot{x} = Ax + \psi(u, y), \quad y = Cx$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -b & 0 & b & 0 \\ 0 & 0 & 0 & 1 \\ c & 0 & -c & 0 \end{bmatrix}, \qquad \psi = \begin{bmatrix} 0 \\ -a \sin y \\ 0 \\ du \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 \end{bmatrix}, \quad (A, C) \text{ is observable}$$

Example 8.16 (Inverted pendulum)

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = a(\sin x_1 + u \cos x_1), \quad y = x_1$$

$$\dot{x} = Ax + \psi(u, y), \quad y = Cx$$

$$A = \left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right], \qquad \psi = \left[\begin{array}{c} 0 \\ a(\sin y + u \cos y) \end{array}\right]$$

$$C = \left[\begin{array}{cc} 1 & 0 \end{array}\right]$$

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i, \qquad y = h(x)$$

Is there $z = T(x)$ such that

$$\dot{z} = A_c z + \phi(y) + \sum_{i=1}^{m} \gamma_i(y)u_i, \qquad y = C_c z$$

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & 0 & 1 \\ 0 & \dots & \dots & 0 & 0 \end{bmatrix}, \qquad C_c = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \end{bmatrix}$$

$$\dot{x} = f(x), \qquad y = h(x)$$

$$\Phi(x) = \begin{bmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{n-1} h(x) \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \\ \vdots \\ y^{(n-1)} \end{bmatrix}$$

$$\tilde{\Phi}(z) = \Phi(x)\big|_{x=T^{-1}(z)} = \begin{bmatrix} z_1 \\ z_2 + F_1(z_1) \\ \vdots \\ z_n + F_{n-1}(z_1, \ldots, z_{n-1}) \end{bmatrix}$$

$$\frac{\partial \tilde{\Phi}}{\partial z} = \frac{\partial \Phi}{\partial x} \frac{\partial T^{-1}}{\partial z}$$

$$\frac{\partial \tilde{\Phi}}{\partial z} = \begin{bmatrix} 1 & 0 & \cdots & & 0 \\ * & 1 & 0 & & 0 \\ \vdots & & & & \vdots \\ * & \cdots & * & 1 & 0 \\ * & & \cdots & * & 1 \end{bmatrix}$$

$\frac{\partial \Phi}{\partial x}$ must be nonsingular

$$\frac{\partial \Phi}{\partial x}\tau = b, \qquad b = \text{col}\begin{pmatrix} 0, & \cdots & 0, & 1 \end{pmatrix}$$

$$L_\tau L_f^k h(x) = 0, \quad 0 \le k \le n-2, \quad L_\tau L_f^{n-1} h(x) = 1$$

Equivalently

$$L_{ad_f^k \tau} h(x) = 0, \quad 0 \le k \le n-2, \quad L_{ad_f^{n-1} \tau} h(x) = (-1)^{n-1}$$

Define $\quad \tau_k = (-1)^{n-k} ad_f^{n-k} \tau, \qquad 1 \le k \le n$

$$\frac{\partial T}{\partial x}\begin{bmatrix} \tau_1 & \tau_2 & \cdots & \tau_n \end{bmatrix} = I$$

$$\frac{\partial T}{\partial x}\tau_k = e_k \stackrel{\text{def}}{=} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{array}{l} \\ \\ \\ \leftarrow k\text{th row} \\ \\ \\ \\ \end{array}$$

$$\begin{aligned} e_k &= (-1)^{n-k}\frac{\partial T}{\partial x}\, ad_f^{n-k}\tau \;=\; (-1)^{n-k}\frac{\partial T}{\partial x}[f, ad_f^{n-k-1}\tau] \\ &= (-1)^{n-k}[\tilde{f}(z), (-1)^{n-k-1}e_{k+1}] \;=\; \frac{\partial \tilde{f}}{\partial z}e_{k+1} \end{aligned}$$

$$\frac{\partial \tilde{f}}{\partial z} = \begin{bmatrix} * & 1 & 0 & \dots & 0 \\ * & 0 & 1 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & 0 & 1 \\ * & \dots & \dots & 0 & 0 \end{bmatrix}$$

By integration

$$\tilde{f}(z) = A_c z + \phi(z_1)$$

$$\tilde{h}(z) = h(T^{-1}(z)), \quad \frac{\partial \tilde{h}}{\partial z} = \frac{\partial h}{\partial x}\frac{\partial T^{-1}}{\partial z}$$

$$\frac{\partial T^{-1}}{\partial z} = \begin{bmatrix} \tau_1 & \tau_2 & \cdots & \tau_n \end{bmatrix}_{x=T^{-1}(z)}$$

$$\frac{\partial \tilde{h}}{\partial z} = \begin{bmatrix} (-1)^{n-1} L_{ad_f^{n-1}\tau} h, & (-1)^{n-2} L_{ad_f^{n-2}\tau} h, & \cdots & L_\tau h \end{bmatrix}$$

$$\frac{\partial \tilde{h}}{\partial z} = \begin{bmatrix} 1, & 0, & \cdots & 0 \end{bmatrix} \quad \Rightarrow \quad \tilde{h} = z_1$$

Theorem 8.3

An n-dimensional single-output (SO) systems

$$\dot{x} = f(x), \qquad y = h(x)$$

is transformable into the observer form if and only if there is a domain D_0 such that $\forall x \in D_0$

$$\text{rank}\left[\frac{\partial \Phi}{\partial x}(x)\right] = n, \quad \Phi = \text{col}\left(\; h, \quad L_f h, \quad \cdots \quad L_f^{n-1} h \; \right)$$

and the unique vector field solution τ of

$$\frac{\partial \Phi}{\partial x}\tau = b, \qquad b = \text{col}\left(\; 0, \quad \cdots \quad 0, \quad 1 \; \right)$$

satisfies $\quad [ad_f^i \tau, ad_f^j \tau] = 0, \qquad 0 \le i, j \le n-1$

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i, \qquad y = h(x)$$

When will $\tilde{g}_i(z) = \left.\dfrac{\partial T}{\partial x} g_i(x)\right|_{x=T^{-1}(z)}$ be independent of z_2 to z_n?

$$\frac{\partial T}{\partial x}[g_i, ad_f^{n-k-1}\tau] = [\tilde{g}_i, (-1)^{n-k-1} e_{k+1}] = (-1)^{n-k} \frac{\partial \tilde{g}_i}{\partial z_{k+1}}$$

$$\frac{\partial \tilde{g}_i}{\partial z_{k+1}} = 0 \;\Leftrightarrow\; [g_i, ad_f^{n-k-1}\tau] = 0$$

Corollary 8.1

Suppose the assumptions of Theorem 8.3 are satisfied. Then, the change of variables $z = T(x)$ transforms the system into the observer form if and only if

$$[g_i, ad_f^k \tau] = 0, , \qquad \text{for } 0 \leq k \leq n-2 \text{ and } 1 \leq i \leq m$$

Moreover, if for some i the foregoing condition is strengthened to

$$[g_i, ad_f^k \tau] = 0, , \qquad \text{for } 0 \leq k \leq n-1$$

then the vector field γ_i is constant

Example 8.17

$$\dot{x} = \begin{bmatrix} \beta_1(x_1) + x_2 \\ f_2(x) \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u, \qquad y = x_1$$

$$\Phi(x) = \begin{bmatrix} h(x) \\ L_f h(x) \end{bmatrix} = \begin{bmatrix} x_1 \\ \beta_1(x_1) + x_2 \end{bmatrix}$$

$$\frac{\partial \Phi}{\partial x} = \begin{bmatrix} 1 & 0 \\ \frac{\partial \beta_1}{\partial x_1} & 1 \end{bmatrix}; \qquad \text{rank} \left[\frac{\partial \Phi}{\partial x}(x) \right] = 2, \ \ \forall \ x$$

$$\frac{\partial \Phi}{\partial x} \, \tau = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \ \ \Rightarrow \ \ \tau = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$ad_f\tau = [f,\tau] = -\frac{\partial f}{\partial x}\tau = -\begin{bmatrix} * & 1 \\ * & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\begin{bmatrix} 1 \\ \frac{\partial f_2}{\partial x_2} \end{bmatrix}$$

$$[\tau, ad_f\tau] = \frac{\partial(ad_f\tau)}{\partial x}\tau = -\begin{bmatrix} 0 & 0 \\ \frac{\partial^2 f_2}{\partial x_1 \partial x_2} & \frac{\partial^2 f_2}{\partial x_2^2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$[\tau, ad_f\tau] = 0 \;\Leftrightarrow\; \frac{\partial^2 f_2}{\partial x_2^2} = 0 \;\Leftrightarrow\; f_2(x) = \beta_2(x_1) + x_2\beta_3(x_1)$$

$$[g,\tau] = 0 \quad (g \text{ and } \tau \text{ are constant vector fields})$$

$$[g, ad_f\tau] = \begin{bmatrix} 0 & 0 \\ -\frac{\partial \beta_3}{\partial x_1} & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = 0 \text{ if } \frac{\partial \beta_3}{\partial x_1} = 0 \text{ or } b_1 = 0$$

$$\tau_1 = (-1)^1 ad_f^1 \tau = -ad_f \tau = \begin{bmatrix} 1 \\ \beta_3(x_1) \end{bmatrix}$$

$$\tau_2 = (-1)^0 ad_f^0 \tau = \tau = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\frac{\partial T}{\partial x} \begin{bmatrix} \tau_1, & \tau_2 \end{bmatrix} = I$$

$$\begin{bmatrix} \frac{\partial T_1}{\partial x_1} & \frac{\partial T_1}{\partial x_2} \\ \\ \frac{\partial T_2}{\partial x_1} & \frac{\partial T_2}{\partial x_2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \beta_3(x_1) & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\frac{\partial T_1}{\partial x_2} = 0 \ \text{ and } \ \frac{\partial T_1}{\partial x_1} = 1 \quad \Rightarrow \quad T_1 = x_1$$

$$\frac{\partial T_2}{\partial x_2} = 1 \;\; \text{and} \;\; \frac{\partial T_2}{\partial x_1} + \beta_3(x_1) = 0$$

$$\Rightarrow \quad T_2(x) = x_2 - \int_0^{x_1} \beta_3(\sigma)\ d\sigma$$

$$\dot{z} = Az + \phi(y) + \gamma(y)u, \qquad y = Cz$$

$$A = \left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right], \quad C = \left[\begin{array}{cc} 1 & 0 \end{array}\right]$$

$$\phi = \left[\begin{array}{c} \beta_1(y) + \int_0^y \beta_3(\sigma)\ d\sigma \\ \beta_2(y) - \beta_1(y)\beta_3(y) \end{array}\right], \qquad \gamma = \left[\begin{array}{c} b_1 \\ b_2 - b_1\beta_3(y) \end{array}\right]$$

Special Case: SISO system

$$\dot{x} = f(x) + g(x)u, \quad y = h(x)$$

Suppose the assumptions of Corollary 8.1 hold with

$$[g, ad_f^k \tau] = 0, , \qquad \text{for } 0 \leq k \leq n-1$$

$$z = T(x) \;\rightarrow\; \dot{z} = A_c z + \phi(y) + \gamma u, \qquad y = C_c z$$

Rel deg $= \rho \;\Leftrightarrow\; \gamma = \left[\; 0, \;\; \ldots\;, \;\; 0, \;\; \gamma_\rho, \;\; \ldots\;, \;\; \gamma_n \;\right]^T, \gamma_\rho \neq 0$

Minimum Phase $\;\Leftrightarrow\; \gamma_\rho s^{n-\rho} + \cdots + \gamma_{n-1}s + \gamma_n$ Hurwitz