Nonlinear Control

Lecture # 23

State Feedback Stabilization

Basic Concepts

We want to stabilize the system

$$\dot{x} = f(x, u)$$

at the equilibrium point $x = x_{\mathrm{ss}}$

Steady-State Problem: Find steady-state control u_{ss} s.t.

$$0 = f(x_{\mathrm{ss}}, u_{\mathrm{ss}})$$

$$x_\delta = x - x_{\mathrm{ss}}, \qquad u_\delta = u - u_{\mathrm{ss}}$$

$$\dot{x}_\delta = f(x_{\mathrm{ss}} + x_\delta, u_{\mathrm{ss}} + u_\delta) \stackrel{\mathrm{def}}{=} f_\delta(x_\delta, u_\delta)$$

$$f_\delta(0, 0) = 0$$

$$u_\delta = \phi(x_\delta) \quad \Rightarrow \quad u = u_{\mathrm{ss}} + \phi(x - x_{\mathrm{ss}})$$

State Feedback Stabilization: Given

$$\dot{x} = f(x, u) \qquad\qquad [f(0, 0) = 0]$$

find

$$u = \phi(x) \qquad\qquad [\phi(0) = 0]$$

s.t. the origin is an asymptotically stable equilibrium point of

$$\dot{x} = f(x, \phi(x))$$

f and ϕ are locally Lipschitz functions

Notions of Stabilization

$$\dot{x} = f(x, u), \qquad u = \phi(x)$$

Local Stabilization: The origin of $\dot{x} = f(x, \phi(x))$ is asymptotically stable (e.g., linearization)

Regional Stabilization: The origin of $\dot{x} = f(x, \phi(x))$ is asymptotically stable and a given region G is a subset of the region of attraction (for all $x(0) \in G$, $\lim_{t\to\infty} x(t) = 0$) (e.g., $G \subset \Omega_c = \{V(x) \le c\}$ where Ω_c is an estimate of the region of attraction)

Global Stabilization: The origin of $\dot{x} = f(x, \phi(x))$ is globally asymptotically stable

Semiglobal Stabilization: The origin of $\dot{x} = f(x, \phi(x))$ is asymptotically stable and $\phi(x)$ can be designed such that any given compact set (no matter how large) can be included in the region of attraction (Typically $u = \phi_p(x)$ is dependent on a parameter p such that for any compact set G, p can be chosen to ensure that G is a subset of the region of attraction)

What is the difference between global stabilization and semiglobal stabilization?

Example 9.1

$$\dot{x} = x^2 + u$$

Linearization:

$$\dot{x} = u, \qquad u = -kx, \quad k > 0$$

Closed-loop system:

$$\dot{x} = -kx + x^2$$

Linearization of the closed-loop system yields $\dot{x} = -kx$. Thus, $u = -kx$ achieves local stabilization

The region of attraction is $\{x < k\}$. Thus, for any set $\{-a \leq x \leq b\}$ with $b < k$, the control $u = -kx$ achieves regional stabilization

The control $u = -kx$ does not achieve global stabilization

But it achieves semiglobal stabilization because any compact set $\{|x| \le r\}$ can be included in the region of attraction by choosing $k > r$

The control

$$u = -x^2 - kx$$

achieves global stabilization because it yields the linear closed-loop system $\dot{x} = -kx$ whose origin is globally exponentially stable

Linear Systems

$$\dot{x} = Ax + Bu$$

(A, B) is stabilizable (controllable or every uncontrollable eigenvalue has a negative real part)

Find K such that $(A - BK)$ is Hurwitz

$$u = -Kx$$

Typical methods:

- Eigenvalue Placement
- Eigenvalue-Eigenvector Placement
- LQR

Linearization

$$\dot{x} = f(x, u)$$

$f(0, 0) = 0$ and f is continuously differentiable in a domain $D_x \times D_u$ that contains the origin $(x = 0,\ u = 0)$ ($D_x \subset R^n$, $D_u \subset R^m$)

$$\dot{x} = Ax + Bu$$

$$A = \left.\frac{\partial f}{\partial x}(x, u)\right|_{x=0,u=0}; \qquad B = \left.\frac{\partial f}{\partial u}(x, u)\right|_{x=0,u=0}$$

Assume (A, B) is stabilizable. Design a matrix K such that $(A - BK)$ is Hurwitz

$$u = -Kx$$

Closed-loop system:

$$\dot{x} = f(x, -Kx)$$

Linearization:

$$\begin{aligned} \dot{x} &= \left[\frac{\partial f}{\partial x}(x, -Kx) + \frac{\partial f}{\partial u}(x, -Kx)\ (-K)\right]_{x=0} x \\ &= (A - BK)x \end{aligned}$$

Since $(A - BK)$ is Hurwitz, the origin is an exponentially stable equilibrium point of the closed-loop system

Example 9.2 (Pendulum Equation)

$$\ddot{\theta} = -\sin\theta - b\dot{\theta} + cu$$

Stabilize the pendulum at $\theta = \delta_1$

$$0 = -\sin\delta_1 + cu_{\rm ss}$$

$$x_1 = \theta - \delta_1, \quad x_2 = \dot{\theta}, \quad u_\delta = u - u_{\rm ss}$$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -[\sin(x_1 + \delta_1) - \sin\delta_1] - bx_2 + cu_\delta \end{aligned}$$

$$A = \left[\begin{array}{cc} 0 & 1 \\ -\cos(x_1+\delta_1) & -b \end{array}\right]_{x_1=0} = \left[\begin{array}{cc} 0 & 1 \\ -\cos\delta_1 & -b \end{array}\right]$$

$$A = \begin{bmatrix} 0 & 1 \\ -\cos\delta_1 & -b \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ c \end{bmatrix}$$

$$K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$$

$$A - BK = \begin{bmatrix} 0 & 1 \\ -(\cos\delta_1 + ck_1) & -(b + ck_2) \end{bmatrix}$$

$$k_1 > -\frac{\cos\delta_1}{c}, \quad k_2 > -\frac{b}{c}$$

$$u = \frac{\sin\delta_1}{c} - Kx = \frac{\sin\delta_1}{c} - k_1(\theta - \delta_1) - k_2\dot{\theta}$$

Feedback Linearization

Consider the nonlinear system

$$\dot{x} = f(x) + G(x)u$$

$$f(0) = 0, \quad x \in R^n, \quad u \in R^m$$

Suppose there is a change of variables $z = T(x)$, defined for all $x \in D \subset R^n$, that transforms the system into the controller form

$$\dot{z} = Az + B[\psi(x) + \gamma(x)u]$$

where (A, B) is controllable and $\gamma(x)$ is nonsingular for all $x \in D$

$$u = \gamma^{-1}(x)[-\psi(x) + v] \quad \Rightarrow \quad \dot{z} = Az + Bv$$

$$v = -Kz$$

Design K such that $(A - BK)$ is Hurwitz

The origin $z = 0$ of the closed-loop system

$$\dot{z} = (A - BK)z$$

is globally exponentially stable

$$u = \gamma^{-1}(x)[-\psi(x) - KT(x)]$$

Closed-loop system in the x-coordinates:

$$\dot{x} = f(x) + G(x)\gamma^{-1}(x)[-\psi(x) - KT(x)] \stackrel{\text{def}}{=} f_c(x)$$

What can we say about the stability of $x = 0$ as an equilibrium point of $\dot{x} = f_c(x)$?

$$z = T(x) \;\Rightarrow\; \frac{\partial T}{\partial x}(x) f_c(x) = (A - BK)T(x)$$

$$\frac{\partial f_c}{\partial x}(0) = J^{-1}(A - BK)J, \qquad J = \frac{\partial T}{\partial x}(0) \text{ (nonsingular)}$$

The origin of $\dot{x} = f_c(x)$ is exponentially stable

Is $x = 0$ globally asymptotically stable? In general No

It is globally asymptotically stable if $T(x)$ is a global diffeomorphism

Estimate of the region of attraction: If $T(x)$ is a diffeomorphism on a domain $D \subset R^n$, the equation $\dot{z} = (A - BK)z$ is valid in the domain $T(D)$

$$P(A - BK) + (A - BK)^T P = -Q, \quad Q = Q^T > 0$$

Estimate in the z-coordinates:

$$\Omega_c = \{z^T P z \le c\} \subset T(D)$$

Estimate in the x-coordinates:

$$T^{-1}(\Omega_c) = \{T^T(x) P T(x) \le c\}$$

Example 9.3 (Recall Example 8.12)

$$\dot{x}_1 = a\sin x_2, \qquad \dot{x}_2 = -x_1^2 + u$$

$$z = T(x) = \begin{bmatrix} x_1 \\ a\sin x_2 \end{bmatrix} \;\Rightarrow\; \dot{z} = \begin{bmatrix} z_2 \\ \sqrt{a^2 - z_2^2}(-z_1^2 + u) \end{bmatrix}$$

$$D = \{|x_2| < \pi/2\}, \quad T(D) = \{|z_2| < a\}$$

$$K = \begin{bmatrix} \sigma^2 & 2\sigma \end{bmatrix}, \; \sigma > 0, \;\; \Rightarrow \;\lambda(A - BK) = -\sigma, \; -\sigma$$

$$P = \begin{bmatrix} 3\sigma^2 & \sigma \\ \sigma & 1 \end{bmatrix}, \; Q = \begin{bmatrix} 2\sigma^3 & 0 \\ 0 & 2\sigma \end{bmatrix}, \; c < \min_{|z_2|=a} z^T P z = \frac{2a^2}{3}$$

$$T^{-1}(\Omega_c) = \{3\sigma^2 x_1^2 + 2\sigma a x_1 \sin x_2 + a^2 \sin^2 x_2 \le c\}$$