Nonlinear Control

Lecture # 24

State Feedback Stabilization

Feedback Lineaization

What information do we need to implement the control

$$u = \gamma^{-1}(x)[-\psi(x) - KT(x)] \text{ ?}$$

What is the effect of uncertainty in ψ, γ, and T?

Let $\hat{\psi}(x)$, $\hat{\gamma}(x)$, and $\hat{T}(x)$ be nominal models of $\psi(x)$, $\gamma(x)$, and $T(x)$

$$u = \hat{\gamma}^{-1}(x)[-\hat{\psi}(x) - K\hat{T}(x)]$$

Closed-loop system:

$$\dot{z} = (A - BK)z + B\Delta(z)$$

$$\dot{z} = (A - BK)z + B\Delta(z) \qquad (*)$$

$$V(z) = z^T P z, \qquad P(A - BK) + (A - BK)^T P = -I$$

Lemma 9.1

Suppose (*) is defined in $D_z \subset R^n$

- If $\|\Delta(z)\| \leq k\|z\| \ \forall \ z \in D_z$, $k < 1/(2\|PB\|)$, then the origin of (*) is exponentially stable. It is globally exponentially stable if $D_z = R^n$
- If $\|\Delta(z)\| \leq k\|z\| + \delta \ \forall \ z \in D_z$ and $B_r \subset D_z$, then there exist positive constants c_1 and c_2 such that if $\delta < c_1 r$ and $z(0) \in \{z^T P z \leq \lambda_{\min}(P) r^2\}$, $\|z(t)\|$ will be ultimately bounded by δc_2. If $D_z = R^n$, $\|z(t)\|$ will be globally ultimately bounded by δc_2 for any $\delta > 0$

Example 9.4 (Pendulum Equation)

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\sin(x_1 + \delta_1) - bx_2 + cu$$

$$u = \left(\frac{1}{c}\right)\left[\sin(x_1 + \delta_1) - (k_1 x_1 + k_2 x_2)\right]$$

$$A - BK = \left[\begin{array}{cc} 0 & 1 \\ -k_1 & -(k_2 + b) \end{array}\right]$$

$$u = \left(\frac{1}{\hat{c}}\right)\left[\sin(x_1 + \delta_1) - (k_1 x_1 + k_2 x_2)\right]$$

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -k_1 x_1 - (k_2 + b)x_2 + \Delta(x)$$

$$\Delta(x) = \left(\frac{c - \hat{c}}{\hat{c}}\right)\left[\sin(x_1 + \delta_1) - (k_1 x_1 + k_2 x_2)\right]$$

$$|\Delta(x)| \leq k\|x\| + \delta, \quad \forall\ x$$

$$k = \left|\frac{c-\hat{c}}{\hat{c}}\right| \left(1 + \sqrt{k_1^2 + k_2^2}\right), \quad \delta = \left|\frac{c-\hat{c}}{\hat{c}}\right| |\sin\delta_1|$$

$$P(A-BK) + (A-BK)^T P = -I, \quad P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$$

$$k < \frac{1}{2\sqrt{p_{12}^2 + p_{22}^2}} \;\Rightarrow\; \text{GUB}$$

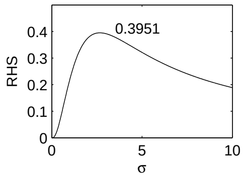
$$\sin\delta_1 = 0 \ \&\ k < \frac{1}{2\sqrt{p_{12}^2 + p_{22}^2}} \;\Rightarrow\; \text{GES}$$

$$b = 0, \quad \hat{c} = m_o/\hat{m}$$

$$k = \Delta_m \left[1 + \sqrt{k_1^2 + k_2^2}\right], \quad \delta = \Delta_m |\sin \delta_1|, \quad \Delta_m = \left|\frac{\hat{m} - m}{m}\right|$$

$$K = \left[\begin{array}{cc} \sigma^2 & 2\sigma \end{array}\right], \ \sigma > 0, \ \Rightarrow \ \lambda(A - BK) = -\sigma, \ -\sigma$$

$$k < \frac{1}{2\sqrt{p_{12}^2 + p_{22}^2}} \Leftrightarrow \Delta_m < \frac{2\sigma^3}{[1 + \sigma\sqrt{\sigma^2 + 4}]\sqrt{4\sigma^2 + (\sigma^2 + 1)^2}}$$



Is feedback linearization a good idea?

Example 9.5

$$\dot{x} = ax - bx^3 + u, \qquad a, b > 0$$

$$u = -(k+a)x + bx^3,\ k > 0, \quad \Rightarrow \quad \dot{x} = -kx$$

$-bx^3$ is a damping term. Why cancel it?

$$u = -(k+a)x,\ k > 0, \quad \Rightarrow \quad \dot{x} = -kx - bx^3$$

Which design is better?

Example 9.6

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -h(x_1) + u$$

$$h(0) = 0 \text{ and } x_1 h(x_1) > 0, \ \forall \ x_1 \neq 0$$

Feedback Linearization:

$$u = h(x_1) - (k_1 x_1 + k_2 x_2)$$

With $y = x_2$, the system is passive with

$$V = \int_0^{x_1} h(z) \ dz + \tfrac{1}{2} x_2^2$$

$$\dot{V} = h(x_1)\dot{x}_1 + x_2 \dot{x}_2 = yu$$

The control

$$u = -\sigma(x_2), \qquad \sigma(0) = 0, \; x_2\sigma(x_2) > 0 \; \forall \; x_2 \neq 0$$

creates a feedback connection of two passive systems with storage function V

$$\dot{V} = -x_2\sigma(x_2)$$

$$x_2(t) \equiv 0 \;\Rightarrow\; \dot{x}_2(t) \equiv 0 \;\Rightarrow\; h(x_1(t)) \equiv 0 \;\Rightarrow\; x_1(t) \equiv 0$$

Asymptotic stability of the origin follows from the invariance principle

Which design is better?

The control $u = -\sigma(x_2)$ has two advantages:

- It does not use a model of h
- The flexibility in choosing σ can be used to reduce $|u|$

However, $u = -\sigma(x_2)$ cannot arbitrarily assign the rate of decay of $x(t)$. Linearization of the closed-loop system at the origin yields the characteristic equation

$$s^2 + \sigma'(0)s + h'(0) = 0$$

One of the two roots cannot be moved to the left of $\mathrm{Re}[s] = -\sqrt{h'(0)}$

Partial Feedback Linearization

Consider the nonlinear system

$$\dot{x} = f(x) + G(x)u \qquad [f(0) = 0]$$

Suppose there is a change of variables

$$z = \begin{bmatrix} \eta \\ \xi \end{bmatrix} = T(x) = \begin{bmatrix} T_1(x) \\ T_2(x) \end{bmatrix}$$

defined for all $x \in D \subset R^n$, that transforms the system into

$$\dot{\eta} = f_0(\eta, \xi), \quad \dot{\xi} = A\xi + B[\psi(x) + \gamma(x)u]$$

(A, B) is controllable and $\gamma(x)$ is nonsingular for all $x \in D$

$$u = \gamma^{-1}(x)[-\psi(x) + v]$$

$$\dot{\eta} = f_0(\eta, \xi), \qquad\qquad \dot{\xi} = A\xi + Bv$$

$v = -K\xi$, where $(A - BK)$ is Hurwitz

Lemma 9.2

The origin of the cascade connection

$$\dot{\eta} = f_0(\eta, \xi), \quad \dot{\xi} = (A - BK)\xi$$

is asymptotically (exponentially) stable if the origin of $\dot{\eta} = f_0(\eta, 0)$ is asymptotically (exponentially) stable

Proof

With $b > 0$ sufficiently small,

$$V(\eta, \xi) = bV_1(\eta) + \sqrt{\xi^T P \xi}, \quad (asymptotic)$$

$$V(\eta, \xi) = bV_1(\eta) + \xi^T P \xi, \quad (exponential)$$

If the origin of $\dot{\eta} = f_0(\eta, 0)$ is globally asymptotically stable, will the origin of

$$\dot{\eta} = f_0(\eta, \xi), \qquad \dot{\xi} = (A - BK)\xi$$

be globally asymptotically stable?

In general No

Example 9.7

The origin of $\dot{\eta} = -\eta$ is globally exponentially stable, but

$$\dot{\eta} = -\eta + \eta^2\xi, \qquad \dot{\xi} = -k\xi, \quad k > 0$$

has a finite region of attraction $\{\eta\xi < 1 + k\}$

Example 9.8

$$\dot{\eta} = -\,\tfrac{1}{2}(1+\xi_2)\eta^3, \qquad \dot{\xi}_1 = \xi_2, \qquad \dot{\xi}_2 = v$$

The origin of $\dot{\eta} = -\,\frac{1}{2}\eta^3$ is globally asymptotically stable

$$v = -k^2\xi_1 - 2k\xi_2 \stackrel{\text{def}}{=} -K\xi \quad \Rightarrow \quad A - BK = \begin{bmatrix} 0 & 1 \\ -k^2 & -2k \end{bmatrix}$$

The eigenvalues of $(A - BK)$ are $-k$ and $-k$

$$e^{(A-BK)t} = \begin{bmatrix} (1+kt)e^{-kt} & te^{-kt} \\ -k^2te^{-kt} & (1-kt)e^{-kt} \end{bmatrix}$$

Peaking Phenomenon:

$$\max_t\{k^2te^{-kt}\} = \frac{k}{e} \to \infty \text{ as } k \to \infty$$

$$\xi_1(0) = 1,\ \xi_2(0) = 0 \quad \Rightarrow \quad \xi_2(t) = -k^2te^{-kt}$$

$$\dot{\eta} = -\ \tfrac{1}{2}\left(1 - k^2te^{-kt}\right)\eta^3, \qquad \eta(0) = \eta_0$$

$$\eta^2(t) = \frac{\eta_0^2}{1 + \eta_0^2[t + (1 + kt)e^{-kt} - 1]}$$

If $\eta_0^2 > 1$, the system will have a finite escape time if k is chosen large enough

Lemma 9.3

The origin of

$$\dot{\eta} = f_0(\eta, \xi), \qquad\qquad \dot{\xi} = (A - BK)\xi$$

is globally asymptotically stable if the system $\dot{\eta} = f_0(\eta, \xi)$ is input-to-state stable

Proof

Apply Lemma 4.6

$$u = \gamma^{-1}(x)[-\psi(x) - KT_2(x)]$$

What is the effect of uncertainty in ψ, γ, and T_2?

Let $\hat{\psi}(x)$, $\hat{\gamma}(x)$, and $\hat{T}_2(x)$ be nominal models of $\psi(x)$, $\gamma(x)$, and $T_2(x)$

$$u = \hat{\gamma}^{-1}(x)[-\hat{\psi}(x) - K\hat{T}_2(x)]$$

$$\dot{\eta} = f_0(\eta, \xi), \qquad \dot{\xi} = (A - BK)\xi + B\Delta(z) \qquad (*)$$

$$\Delta = \psi + \gamma\hat{\gamma}^{-1}[-\hat{\psi} - K\hat{T}_2] + KT_2$$

$$\dot{\eta} = f_0(\eta, \xi), \qquad \dot{\xi} = (A - BK)\xi + B\Delta(z) \qquad (*)$$

Lemma 9.4

If $\dot{\eta} = f_0(\eta, \xi)$ is input-to-state stable and $\|\Delta(z)\| \leq \delta$ for all z, for some $\delta > 0$, then the solution of (*) is globally ultimately bounded by a class $\mathcal{K}$ function of δ

Lemma 9.5

If the origin of $\dot{\eta} = f_0(\eta, 0)$ is exponentially stable, (*) is defined in $D_z \subset R^n$, and $\|\Delta(z)\| \leq k\|z\| + \delta \ \forall \ z \in D_z$, then there exist a neighborhood N_z of $z = 0$ and positive constants k^*, δ^*, and c such that for $k < k^*$, $\delta < \delta^*$, and $z(0) \in N_z$, $\|z(t)\|$ will be ultimately bounded by $c\delta$. If $\delta = 0$, the origin of (*) will be exponentially stable