Nonlinear Control

Lecture # 25

State Feedback Stabilization

Backstepping

$$\begin{aligned}\dot{\eta} &= f_a(\eta) + g_a(\eta)\xi \\ \dot{\xi} &= f_b(\eta,\xi) + g_b(\eta,\xi)u, \quad g_b \neq 0, \quad \eta \in R^n, \ \xi, \ u \in R\end{aligned}$$

Stabilize the origin using state feedback

View ξ as "virtual" control input to the system

$$\dot{\eta} = f_a(\eta) + g_a(\eta)\xi$$

Suppose there is $\xi = \phi(\eta)$ that stabilizes the origin of

$$\dot{\eta} = f_a(\eta) + g_a(\eta)\phi(\eta)$$

$$\frac{\partial V_a}{\partial \eta}[f_a(\eta) + g_a(\eta)\phi(\eta)] \leq -W(\eta)$$

$$z = \xi - \phi(\eta)$$

$$\begin{aligned}
\dot{\eta} &= [f_a(\eta) + g_a(\eta)\phi(\eta)] + g_a(\eta)z \\
\dot{z} &= F(\eta, \xi) + g_b(\eta, \xi)u
\end{aligned}$$

$$V(\eta, \xi) = V_a(\eta) + \tfrac{1}{2}z^2 = V_a(\eta) + \tfrac{1}{2}[\xi - \phi(\eta)]^2$$

$$\begin{aligned}
\dot{V} &= \frac{\partial V_a}{\partial \eta}[f_a(\eta) + g_a(\eta)\phi(\eta)] + \frac{\partial V_a}{\partial \eta}g_a(\eta)z \\
&\quad + zF(\eta, \xi) + zg_b(\eta, \xi)u \\
&\le -W(\eta) + z\left[\frac{\partial V_a}{\partial \eta}g_a(\eta) + F(\eta, \xi) + g_b(\eta, \xi)u\right]
\end{aligned}$$

$$\dot{V} \le -W(\eta) + z\left[\frac{\partial V_a}{\partial \eta} g_a(\eta) + F(\eta,\xi) + g_b(\eta,\xi)u\right]$$

$$u = -\,\frac{1}{g_b(\eta,\xi)}\left[\frac{\partial V_a}{\partial \eta} g_a(\eta) + F(\eta,\xi) + kz\right], \quad k > 0$$

$$\dot{V} \le -W(\eta) - kz^2$$

Example 9.9

$$\dot{x}_1 = x_1^2 - x_1^3 + x_2, \qquad \dot{x}_2 = u$$

$$\dot{x}_1 = x_1^2 - x_1^3 + x_2$$

$$x_2 = \phi(x_1) = -x_1^2 - x_1 \;\;\Rightarrow\;\; \dot{x}_1 = -x_1 - x_1^3$$

$$V_a(x_1) = \tfrac{1}{2}x_1^2 \;\;\Rightarrow\;\; \dot{V}_a = -x_1^2 - x_1^4, \quad \forall\; x_1 \in R$$

$$z_2 = x_2 - \phi(x_1) = x_2 + x_1 + x_1^2$$

$$\begin{aligned} \dot{x}_1 &= -x_1 - x_1^3 + z_2 \\ \dot{z}_2 &= u + (1 + 2x_1)(-x_1 - x_1^3 + z_2) \end{aligned}$$

$$V(x) = \tfrac{1}{2}x_1^2 + \tfrac{1}{2}z_2^2$$

$$\begin{aligned} \dot{V} &= x_1(-x_1 - x_1^3 + z_2) \\ &\quad + z_2[u + (1 + 2x_1)(-x_1 - x_1^3 + z_2)] \end{aligned}$$

$$\begin{aligned} \dot{V} &= -x_1^2 - x_1^4 \\ &\quad + z_2[x_1 + (1 + 2x_1)(-x_1 - x_1^3 + z_2) + u] \end{aligned}$$

$$u = -x_1 - (1 + 2x_1)(-x_1 - x_1^3 + z_2) - z_2$$

$$\dot{V} = -x_1^2 - x_1^4 - z_2^2$$

The origin is globally asymptotically stable

Example 9.10

$$\dot{x}_1 = x_1^2 - x_1^3 + x_2, \qquad \dot{x}_2 = x_3, \qquad \dot{x}_3 = u$$

$$\dot{x}_1 = x_1^2 - x_1^3 + x_2, \qquad \dot{x}_2 = x_3$$

$$x_3 = -x_1 - (1 + 2x_1)(-x_1 - x_1^3 + z_2) - z_2 \stackrel{\text{def}}{=} \phi(x_1, x_2)$$

$$V_a(x) = \tfrac{1}{2}x_1^2 + \tfrac{1}{2}z_2^2, \qquad \dot{V}_a = -x_1^2 - x_1^4 - z_2^2$$

$$z_3 = x_3 - \phi(x_1, x_2)$$

$$\begin{aligned} \dot{x}_1 &= x_1^2 - x_1^3 + x_2, \qquad \dot{x}_2 = \phi(x_1, x_2) + z_3 \\ \dot{z}_3 &= u - \frac{\partial \phi}{\partial x_1}(x_1^2 - x_1^3 + x_2) - \frac{\partial \phi}{\partial x_2}(\phi + z_3) \end{aligned}$$

$$V = V_a + \tfrac{1}{2}z_3^2$$

$$\begin{aligned}\dot{V} &= \frac{\partial V_a}{\partial x_1}(x_1^2 - x_1^3 + x_2) + \frac{\partial V_a}{\partial x_2}(z_3 + \phi) \\ &\quad + z_3\left[u - \frac{\partial \phi}{\partial x_1}(x_1^2 - x_1^3 + x_2) - \frac{\partial \phi}{\partial x_2}(z_3 + \phi)\right]\end{aligned}$$

$$\begin{aligned}\dot{V} &= -x_1^2 - x_1^4 - (x_2 + x_1 + x_1^2)^2 \\ &\quad + z_3\left[\frac{\partial V_a}{\partial x_2} - \frac{\partial \phi}{\partial x_1}(x_1^2 - x_1^3 + x_2) - \frac{\partial \phi}{\partial x_2}(z_3 + \phi) + u\right]\end{aligned}$$

$$u = -\frac{\partial V_a}{\partial x_2} + \frac{\partial \phi}{\partial x_1}(x_1^2 - x_1^3 + x_2) + \frac{\partial \phi}{\partial x_2}(z_3 + \phi) - z_3$$

The origin is globally asymptotically stable

Strict-Feedback Form

$$
\begin{aligned}
\dot{x} &= f_0(x) + g_0(x) z_1 \\
\dot{z}_1 &= f_1(x, z_1) + g_1(x, z_1) z_2 \\
\dot{z}_2 &= f_2(x, z_1, z_2) + g_2(x, z_1, z_2) z_3 \\
&\vdots \\
\dot{z}_{k-1} &= f_{k-1}(x, z_1, \ldots, z_{k-1}) + g_{k-1}(x, z_1, \ldots, z_{k-1}) z_k \\
\dot{z}_k &= f_k(x, z_1, \ldots, z_k) + g_k(x, z_1, \ldots, z_k) u
\end{aligned}
$$

$$
g_i(x, z_1, \ldots, z_i) \neq 0 \quad \text{for } 1 \leq i \leq k
$$

Example 9.12

$$\dot{x} = -x + x^2 z, \qquad \dot{z} = u$$

$$\dot{x} = -x + x^2 z$$

$$z = 0 \;\Rightarrow\; \dot{x} = -x, \quad V_a = \tfrac{1}{2}x^2 \;\Rightarrow\; \dot{V}_a = -x^2$$

$$V = \tfrac{1}{2}(x^2 + z^2)$$

$$\dot{V} = x(-x + x^2 z) + zu = -x^2 + z(x^3 + u)$$

$$u = -x^3 - kz,\; k > 0,\; \Rightarrow\; \dot{V} = -x^2 - kz^2$$

Global stabilization

Compare with semiglobal stabilization in Example 9.7

Example 9.13

$$\dot{x} = x^2 - xz, \qquad \dot{z} = u$$

$$\dot{x} = x^2 - xz$$

$$z = x + x^2 \;\Rightarrow\; \dot{x} = -x^3, \quad V_0(x) = \frac{1}{2}x^2 \;\Rightarrow\; \dot{V} = -x^4$$

$$V = V_0 + \frac{1}{2}(z - x - x^2)^2$$

$$\dot{V} = -x^4 + (z - x - x^2)[-x^2 + u - (1 + 2x)(x^2 - xz)]$$

$$u = (1 + 2x)(x^2 - xz) + x^2 - k(z - x - x^2), \; k > 0$$

$$\dot{V} = -x^4 - k(z - x - x^2)^2 \qquad \text{Global stabilization}$$

Passivity-Based Control

$$\dot{x} = f(x, u), \qquad y = h(x), \qquad f(0, 0) = 0$$

$$u^T y \geq \dot{V} = \frac{\partial V}{\partial x} f(x, u)$$

Theorem 9.1

If the system is

(1) passive with a radially unbounded positive definite storage function and

(2) zero-state observable,

then the origin can be globally stabilized by

$$u = -\phi(y), \qquad \phi(0) = 0, \quad y^T \phi(y) > 0 \ \forall \ y \neq 0$$

Proof

$$\dot{V} = \frac{\partial V}{\partial x} f(x, -\phi(y)) \leq -y^T \phi(y) \leq 0$$

$$\dot{V}(x(t)) \equiv 0 \;\Rightarrow\; y(t) \equiv 0 \;\Rightarrow\; u(t) \equiv 0 \;\Rightarrow\; x(t) \equiv 0$$

Apply the invariance principle

A given system may be made passive by

(1) Choice of output,
(2) Feedback,

or both

Choice of Output

$$\dot{x} = f(x) + G(x)u, \qquad \frac{\partial V}{\partial x} f(x) \le 0, \quad \forall\ x$$

No output is defined. Choose the output as

$$y = h(x) \stackrel{\text{def}}{=} \left[\frac{\partial V}{\partial x} G(x)\right]^T$$

$$\dot{V} = \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} G(x) u \le y^T u$$

Check zero-state observability

Example 9.14

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1^3 + u$$

$$V(x) = \tfrac{1}{4}x_1^4 + \tfrac{1}{2}x_2^2$$

With $u = 0 \qquad \dot{V} = x_1^3 x_2 - x_2 x_1^3 = 0$

Take $y = \dfrac{\partial V}{\partial x} G = \dfrac{\partial V}{\partial x_2} = x_2$

Is it zero-state observable?

with $u = 0$, $\qquad y(t) \equiv 0 \;\; \Rightarrow \;\; x(t) \equiv 0$

$u = -kx_2$ or $u = -(2k/\pi)\tan^{-1}(x_2) \qquad (k > 0)$

Feedback Passivation

Definition

The system

$$\dot{x} = f(x) + G(x)u, \quad y = h(x) \quad (*)$$

is equivalent to a passive system if $\exists\ u = \alpha(x) + \beta(x)v$ such that

$$\dot{x} = f(x) + G(x)\alpha(x) + G(x)\beta(x)v, \qquad y = h(x)$$

is passive

Theorem [20]

The system (*) is locally equivalent to a passive system (with a positive definite storage function) if it has relative degree one at $x = 0$ and the zero dynamics have a stable equilibrium point at the origin with a positive definite Lyapunov function

Example 9.15 (m-link Robot Manipulator)

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + D\dot{q} + g(q) = u$$

$$M = M^T > 0,\ (\dot{M} - 2C)^T = -(\dot{M} - 2C),\ D = D^T \geq 0$$

Stabilize the system at $q = q_r$

$$e = q - q_r, \qquad \dot{e} = \dot{q}$$

$$M(q)\ddot{e} + C(q,\dot{q})\dot{e} + D\dot{e} + g(q) = u$$

$(e = 0, \dot{e} = 0)$ is not an open-loop equilibrium point

$$u = g(q) - K_p e + v, \qquad (K_p = K_p^T > 0)$$

$$M(q)\ddot{e} + C(q,\dot{q})\dot{e} + D\dot{e} + K_p e = v$$

$$M(q)\ddot{e} + C(q,\dot{q})\dot{e} + D\dot{e} + K_p e = v$$

$$V = \tfrac{1}{2}\dot{e}^T M(q)\dot{e} + \tfrac{1}{2}e^T K_p e$$

$$\dot{V} = \tfrac{1}{2}\dot{e}^T(\dot{M} - 2C)\dot{e} - \dot{e}^T D\dot{e} - \dot{e}^T K_p e + \dot{e}^T v + e^T K_p \dot{e} \le \dot{e}^T v$$

$$y = \dot{e}$$

Is it zero-state observable? Set $v = 0$

$$\dot{e}(t) \equiv 0 \;\Rightarrow\; \ddot{e}(t) \equiv 0 \;\Rightarrow\; K_p e(t) \equiv 0 \;\Rightarrow\; e(t) \equiv 0$$

$$v = -\phi(\dot{e}), \qquad [\phi(0) = 0,\; \dot{e}^T\phi(\dot{e}) > 0, \;\; \forall \dot{e} \neq 0]$$

$$u = g(q) - K_p e - \phi(\dot{e})$$

Special case: $\quad u = g(q) - K_p e - K_d\dot{e}, \qquad K_d = K_d^T > 0$