Nonlinear Control

Lecture # 26

State Feedback Stabilization

Passivity-Based Control: Cascade Connection

$$\dot{x} = f_a(x) + F(x,y)y, \qquad \dot{z} = f(z) + G(z)u, \qquad y = h(z)$$

$$f_a(0) = 0, \qquad f(0) = 0, \qquad h(0) = 0$$

$$\frac{\partial V}{\partial z} f(z) + \frac{\partial V}{\partial z} G(z) u \le y^T u, \qquad \frac{\partial W}{\partial x} f_a(x) \le 0$$

$$U(x,z) = W(x) + V(z)$$

$$\dot{U} \le \frac{\partial W}{\partial x} F(x,y) y + y^T u \;=\; y^T \left[u + \left(\frac{\partial W}{\partial x} F(x,y) \right)^T \right]$$

$$u = -\left(\frac{\partial W}{\partial x} F(x,y) \right)^T + v \;\;\Rightarrow\;\; \dot{U} \le y^T v$$

The system

$$
\begin{aligned}
\dot{x} &= f_a(x) + F(x,y)y \\
\dot{z} &= f(z) - G(z)\left(\frac{\partial W}{\partial x}F(x,y)\right)^T + G(z)v \\
y &= h(z)
\end{aligned}
$$

with input v and output y is passive with storage function U

$$v = -\phi(y), \quad [\phi(0) = 0,\ y^T\phi(y) > 0\ \forall\ y \neq 0]$$

$$\dot{U} \leq \frac{\partial W}{\partial x} f_a(x) - y^T\phi(y) \leq 0, \dot{U} = 0 \Rightarrow x = 0 \& y = 0 \Rightarrow u = 0$$

ZSO of driving system: $\quad \dot{U}(t) \equiv 0 \ \Rightarrow \ z(t) \equiv 0$

Theorem 9.2

Suppose

- the system

$$\dot{z} = f(z) + G(z)u, \qquad y = h(z)$$

  is zero-state observable and passive with a radially unbounded, positive definite storage function;
- the origin of $\dot{x} = f_a(x)$ is globally asymptotically stable and $W(x)$ is a radially unbounded, positive definite Lyapunov function

Then, $\quad u = -\left(\frac{\partial W}{\partial x} F(x, y)\right)^T - \phi(y),$

globally stabilizes the origin $(x = 0, z = 0)$

Example 9.16 (see Examples 9.7 and 9.12)

$$\dot{x} = -x + x^2 z, \qquad \dot{z} = u$$

With $y = z$ as the output, the system takes the form of the cascade connection

$$\dot{z} = u, \quad y = z$$

is passive with $V(z) = \frac{1}{2}z^2$ and zero-state observable

$$\dot{x} = -x, \quad W(x) = \frac{1}{2}x^2 \;\Rightarrow\; \dot{W} = -x^2$$

$$u = -x^3 - kz, \quad k > 0$$

Control Lyapunov Functions

$$\dot{x} = f(x) + g(x)u, \qquad f(0) = 0,\ x \in R^n,\ u \in R$$

Suppose there is a continuous stabilizing state feedback control $u = \chi(x)$ such that the origin of

$$\dot{x} = f(x) + g(x)\chi(x)$$

is asymptotically stable

By the converse Lyapunov theorem, there is $V(x)$ such that

$$\frac{\partial V}{\partial x}[f(x) + g(x)\chi(x)] < 0, \qquad \forall\ x \in D,\ x \neq 0$$

If $u = \chi(x)$ is globally stabilizing, then $D = R^n$ and $V(x)$ is radially unbounded

$$\frac{\partial V}{\partial x}[f(x) + g(x)\chi(x)] < 0, \qquad \forall\ x \in D,\ x \neq 0$$

$$\frac{\partial V}{\partial x} g(x) = 0 \text{ for } x \in D,\ x \neq 0 \ \ \Rightarrow \ \ \frac{\partial V}{\partial x} f(x) < 0$$

Definition

A continuously differentiable positive definite function $V(x)$ is a *Control Lyapunov Function* (CLF) for the system $\dot{x} = f(x) + g(x)u$ if

$$\frac{\partial V}{\partial x} g(x) = 0 \text{ for } x \in D,\ x \neq 0 \ \ \Rightarrow \ \ \frac{\partial V}{\partial x} f(x) < 0 \qquad (*)$$

It is a *Global Control Lyapunov Function* if it is radially unbounded and $(*)$ holds with $D = R^n$

The system $\dot{x} = f(x) + g(x)u$ is stabilizable by a state feedback control only if it has a CLF

Is it sufficient? Yes

Sontag's Formula:

$$\phi(x) = \begin{cases} -\dfrac{\frac{\partial V}{\partial x} f + \sqrt{\left(\frac{\partial V}{\partial x} f\right)^2 + \left(\frac{\partial V}{\partial x} g\right)^4}}{\left(\frac{\partial V}{\partial x} g\right)}, & \text{if } \frac{\partial V}{\partial x} g \neq 0 \\[2ex] 0, & \text{if } \frac{\partial V}{\partial x} g = 0 \end{cases}$$

$$\dot{x} = f(x) + g(x)\phi(x)$$

$$\dot{V} = \frac{\partial V}{\partial x}[f(x) + g(x)\phi(x)]$$

If $x \neq 0$ and $\dfrac{\partial V}{\partial x}g(x) = 0, \qquad \dot{V} = \dfrac{\partial V}{\partial x}f(x) < 0$

If $x \neq 0$ and $\dfrac{\partial V}{\partial x}g(x) \neq 0$

$$\begin{aligned} \dot{V} &= \tfrac{\partial V}{\partial x}f - \left[\tfrac{\partial V}{\partial x}f + \sqrt{\left(\tfrac{\partial V}{\partial x}f\right)^2 + \left(\tfrac{\partial V}{\partial x}g\right)^4}\right] \\ &= -\sqrt{\left(\tfrac{\partial V}{\partial x}f\right)^2 + \left(\tfrac{\partial V}{\partial x}g\right)^4} < 0 \end{aligned}$$

Lemma 9.6

If $f(x)$, $g(x)$ and $V(x)$ are smooth then $\phi(x)$ will be smooth for $x \neq 0$. If they are of class $\mathcal{C}^{\ell+1}$ for $\ell \geq 1$, then $\phi(x)$ will be of class $\mathcal{C}^{\ell}$. Continuity at $x = 0$:

- $\phi(x)$ is continuous at $x = 0$ if $V(x)$ has the small control property; namely, given any $\varepsilon > 0$ there $\delta > 0$ such that if $x \neq 0$ and $\|x\| < \delta$, then there is u with $\|u\| < \varepsilon$ such that
$$\frac{\partial V}{\partial x}[f(x) + g(x)u] < 0$$
- $\phi(x)$ is locally Lipschitz at $x = 0$ if there is a locally Lipschitz function $\chi(x)$, with $\chi(0) = 0$, such that
$$\frac{\partial V}{\partial x}[f(x) + g(x)\chi(x)] < 0, \quad \text{for } x \neq 0$$

How can we find a CLF?

If we know of any stabilizing control with a corresponding Lyapunov function V, then V is a CLF

- Feedback Linearization

$$\dot{x} = f(x) + G(x)u, \quad z = T(x), \quad \dot{z} = (A - BK)z$$

$$P(A - BK) + (A - BK)^T P = -Q, \quad Q = Q^T > 0$$

$$V = z^T P z = T^T(x) P T(x) \text{ is a CLF}$$

- Backstepping

Example 9.17

$$\dot{x} = x - x^3 + u$$

Feedback Linearization:

$$u = \chi(x) = -x + x^3 - \alpha x \qquad (\alpha > 0)$$

$$\dot{x} = -\alpha x$$

$$V(x) = \tfrac{1}{2}x^2 \text{ is a CLF}$$

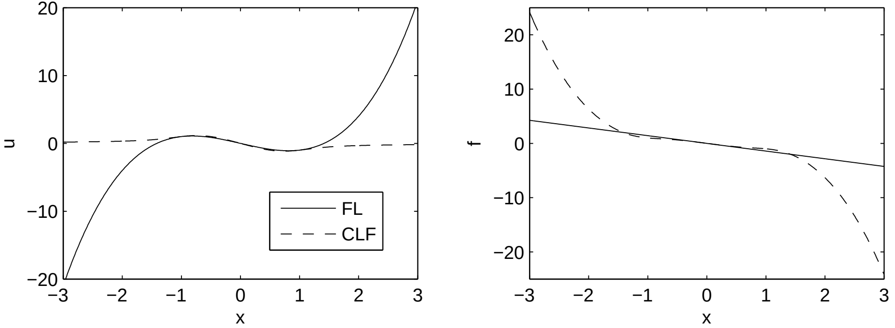
$$\frac{\partial V}{\partial x} g = x, \qquad \frac{\partial V}{\partial x} f = x(x - x^3)$$

$$
\begin{aligned}
& -\frac{\frac{\partial V}{\partial x}f + \sqrt{\left(\frac{\partial V}{\partial x}f\right)^2 + \left(\frac{\partial V}{\partial x}g\right)^4}}{\left(\frac{\partial V}{\partial x}g\right)} \\
&= \; -\frac{x(x - x^3) + \sqrt{x^2(x - x^3)^2 + x^4}}{x} \\
&= \; -x + x^3 - x\sqrt{(1 - x^2)^2 + 1}
\end{aligned}
$$

$$
\phi(x) = -x + x^3 - x\sqrt{(1 - x^2)^2 + 1}
$$

Compare with

$$
\chi(x) = -x + x^3 - \alpha x
$$



$$\alpha = \sqrt{2}$$

Robustness Property

Lemma 9.7

Suppose f, g, and V satisfy the conditions of Lemma 9.6 and ϕ is given by Sontag's formula. Then, the origin of $\dot{x} = f(x) + g(x)k\phi(x)$ is asymptotically stable for all $k \geq \frac{1}{2}$. If V is a global control Lyapunov function, then the origin is globally asymptotically stable

Proof

Let

$$q(x) = \tfrac{1}{2}\left[-\frac{\partial V}{\partial x}f + \sqrt{\left(\frac{\partial V}{\partial x}f\right)^2 + \left(\frac{\partial V}{\partial x}g\right)^4}\right]$$

Because $V(x)$ is positive definite and smooth,

$$\frac{\partial V}{\partial x}(0) = 0 \;\Rightarrow\; q(0) = 0$$

For $x \neq 0$

$$\frac{\partial V}{\partial x}g \neq 0 \;\Rightarrow\; q > 0 \;\;\&\;\; \frac{\partial V}{\partial x}g = 0 \;\Rightarrow\; q = -\frac{\partial V}{\partial x}f > 0$$

$q(x)$ is positive definite

$$u = k\phi(x) \quad \Rightarrow \quad \dot{x} = f(x) + g(x)k\phi(x)$$

$$\dot{V} = \frac{\partial V}{\partial x} f + \frac{\partial V}{\partial x} g k \phi$$

For $x \neq 0, \quad \dfrac{\partial V}{\partial x} g = 0 \quad \Rightarrow \quad \dot{V} = \dfrac{\partial V}{\partial x} f < 0$

$$\frac{\partial V}{\partial x} g \neq 0, \qquad \dot{V} = -q + q + \frac{\partial V}{\partial x} f + \frac{\partial V}{\partial x} g k \phi$$

$$\begin{aligned} & q + \frac{\partial V}{\partial x} f + \frac{\partial V}{\partial x} g k \phi \\ & \quad = \; -\left(k - \tfrac{1}{2}\right) \left[\frac{\partial V}{\partial x} f + \sqrt{\left(\frac{\partial V}{\partial x} f\right)^2 + \left(\frac{\partial V}{\partial x} g\right)^4} \; \right] \leq 0 \end{aligned}$$

Example 9.18

Reconsider $\dot{x} = x - x^3 + u$. Compare $u = \chi(x)$ with $u = \phi(x)$

By Lemma 9.7 the origin of $\dot{x} = x - x^3 + k\phi(x)$ is globally asymptotically stable for all $k \geq \frac{1}{2}$

$$\dot{x} = x - x^3 + k\chi(x) = -[k(1+\alpha) - 1]x + (k-1)x^3$$

The origin is not globally asymptotically stable for any $k > 1$

It is exponentially stable for $k > 1/(1+\alpha)$

Region of attraction:

$$\left\{ |x| < \sqrt{1 + \frac{k\alpha}{(k-1)}}) \right\} \rightarrow \left\{ |x| < \sqrt{1+\alpha} \right\} \text{ as } k \rightarrow \infty$$