Nonlinear Control

Lecture # 27

Robust State Feedback Stabilization

Definition 10.1

The system

$$\dot{x} = f(x, u) + \delta(t, x, u)$$

is said to be practically stabilizable by the feedback control $u = \phi(x)$ if for every $b > 0$, the control can be designed such that the solutions are ultimately bounded by b; that is

$$\|x(t)\| \leq b, \qquad \forall \; t \geq T, \quad \text{for some } T > 0$$

- local practical stabilization
- regional practical stabilization
- global practical stabilization
- semiglobal practical stabilization

Sliding Mode Control

Example 10.1

$$\dot{x}_1 = x_2 \qquad \dot{x}_2 = h(x) + g(x)u, \quad g(x) \geq g_0 > 0$$

Sliding Manifold (Surface):

$$s = ax_1 + x_2 = 0$$

$$s(t) \equiv 0 \;\; \Rightarrow \;\; \dot{x}_1 = -ax_1$$

$$a > 0 \;\; \Rightarrow \;\; \lim_{t\to\infty} x_1(t) = 0$$

How can we bring the trajectory to the manifold $s = 0$?

How can we maintain it there?

$$\dot{s} = a\dot{x}_1 + \dot{x}_2 = ax_2 + h(x) + g(x)u$$

Suppose

$$\left| \frac{ax_2 + h(x)}{g(x)} \right| \le \varrho(x)$$

$$V = \tfrac{1}{2}s^2$$

$$\dot{V} = s\dot{s} = s[ax_2 + h(x)] + g(x)su \le g(x)|s|\varrho(x) + g(x)su$$

$$\beta(x) \ge \varrho(x) + \beta_0, \qquad \beta_0 > 0$$

$$s > 0, \qquad u = -\beta(x)$$

$$\dot{V} \le g(x)|s|\varrho(x) - g(x)\beta(x)|s| \le -g(x)\beta_0|s|$$

$$s < 0, \qquad u = \beta(x)$$

$$\dot{V} \le g(x)|s|\varrho(x) + g(x)su = g(x)|s|\varrho(x) - g(x)\beta(x)|s|$$

$$\dot{V} \le g(x)|s|\varrho(x) - g(x)(\varrho(x) + \beta_0)|s| = -g(x)\beta_0|s|$$

$$\mathrm{sgn}(s) = \begin{cases} 1, & s > 0 \\ -1, & s < 0 \end{cases}$$

$$u = -\beta(x)\ \mathrm{sgn}(s) \ \Rightarrow\ \dot{V} \le -g(x)\beta_0|s| \le -g_0\beta_0|s|$$

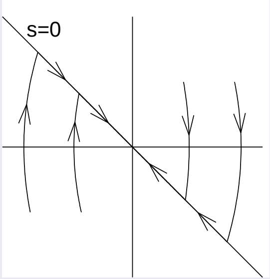
$$\dot{V} \le -g_0\beta_0\sqrt{2V}$$

$$\frac{dV}{\sqrt{V}} \le -g_0\beta_0\sqrt{2}\ dt \ \Rightarrow\ \sqrt{V(s(t))} \le \sqrt{V(s(0))} - g_0\beta_0\frac{1}{\sqrt{2}}\ t$$

$$|s(t)| \le |s(0)| - g_0\beta_0 \; t$$

$s(t)$ reaches zero in finite time

The trajectory cannot leave the surface $s = 0$



Pendulum Equation

$$\ddot{\theta} + \sin\theta + b\dot{\theta} = cu$$

$$0 \le b \le 0.2, \quad 0.5 \le c \le 2$$

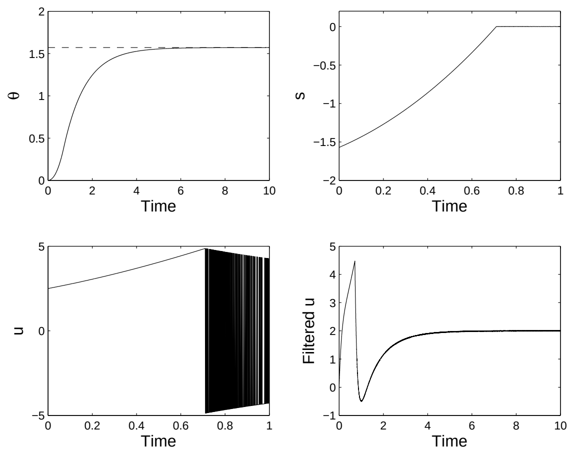
Stabilize the pendulum at $\theta = \pi/2$

$$x_1 = \theta - \frac{\pi}{2},\ x_2 = \dot{\theta} \quad \Rightarrow \quad \dot{x}_1 = x_2, \quad \dot{x}_2 = -\cos x_1 - bx_2 + cu$$

$$s = x_1 + x_2, \quad \dot{s} = x_2 - \cos x_1 - bx_2 + cu$$

$$\left| \frac{x_2 - \cos x_1 - bx_2}{c} \right| = \left| \frac{(1-b)x_2 - \cos x_1}{c} \right| \le 2(|x_2| + 1)$$

$$u = -(2.5 + 2|x_2|)\ \mathrm{sgn}(s)$$



$$b = 0.01,\ c = 0.5,\ \theta(0) = \dot{\theta}(0) = 0$$

Estimate the region of attraction when

$$\left|\frac{ax_2 + h(x)}{g(x)}\right| \leq \varrho(x), \quad \forall x \in D \subset R^2$$

$$\dot{x}_1 = -ax_1 + s \quad \dot{s} = ax_2 + h(x) - g(x)\beta(x)\text{sgn}(s)$$

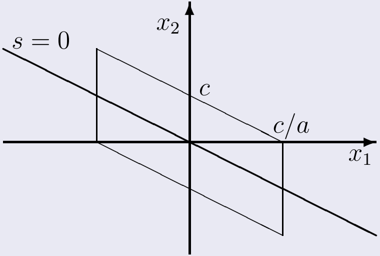
$$s\dot{s} \leq -g_0\beta_0|s| \;\Rightarrow\; \{|s| \leq c\} \text{ is positively invariant}$$

$$V_0 = \tfrac{1}{2}x_1^2 \;\Rightarrow\; \dot{V}_0 = x_1\dot{x}_1 = -ax_1^2 + x_1 s \leq -ax_1^2 + |x_1|c$$

$$\dot{V}_0 \leq 0, \quad \forall\ |s| \leq c \text{ and } |x_1| \geq \frac{c}{a}$$

$$\Omega = \left\{|x_1| \leq \frac{c}{a},\ |s| \leq c\right\} \subset D \text{ is positively invariant}$$

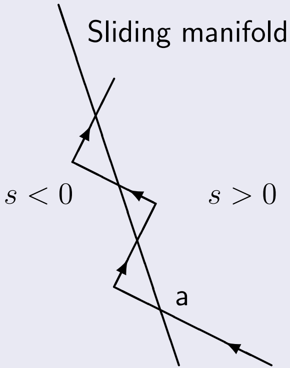
$$\Omega = \left\{ |x_1| \leq \frac{c}{a}, \ |s| \leq c \right\}$$



$$\left| \frac{a x_2 + h(x)}{g(x)} \right| \leq k_1 < k, \quad \forall \ x \in \Omega$$

$$u = -k \ \mathrm{sgn}(s)$$

Chattering



How can we reduce or eliminate chattering?

Reduce the amplitude of the signum function

$$\dot{s} = ax_2 + h(x) + g(x)u$$

$$u = -\ \frac{[ax_2 + \hat{h}(x)]}{\hat{g}(x)} + v$$

$$\dot{s} = \delta(x) + g(x)v$$

$$\delta(x) = a\left[1 - \frac{g(x)}{\hat{g}(x)}\right]x_2 + h(x) -\ \frac{g(x)}{\hat{g}(x)}\hat{h}(x)$$

$$\left|\frac{\delta(x)}{g(x)}\right| \le \varrho(x), \qquad \beta(x) \ge \varrho(x) + \beta_0$$

$$v = -\beta(x)\ \mathrm{sgn}(s)$$

Back to the pendulum equation

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\cos x_1 - bx_2 + cu, \qquad \hat{b} = 0, \hat{c}$$

$$u = \frac{-x_2 + \cos x_1}{\hat{c}} + v \;\Rightarrow\; \dot{s} = \delta + cv$$

$$\frac{\delta}{c} = \left(\frac{1-b}{c} - \frac{1}{\hat{c}}\right) x_2 - \left(\frac{1}{c} - \frac{1}{\hat{c}}\right)\cos x_1$$

Take $\hat{c} = 1/1.2$ to minimize $|(1-b)/c - 1/\hat{c}|$

$$\left|\frac{\delta}{c}\right| \leq 0.8|x_2| + 0.8$$

$$u = 1.2\cos x_1 - 1.2x_2 - (1 + 0.8|x_2|)\,\mathrm{sgn}(s)$$

Simulation with unmodeled actuator dynamics

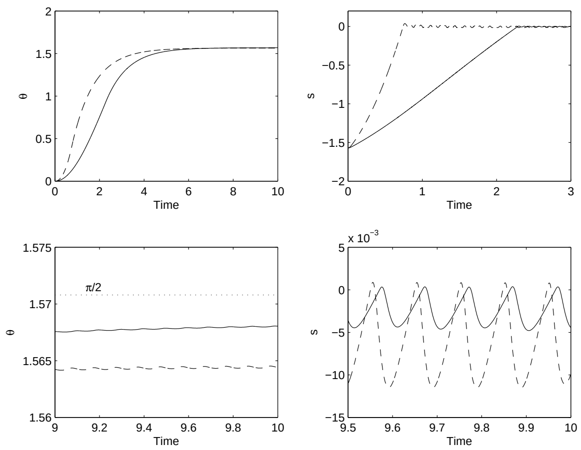
$$\frac{1}{(0.01s+1)^2}$$

Dashed lines:

$$u = -(2.5 + 2|x_2|)\ \mathrm{sgn}(s)$$

Solid lines:

$$u = 1.2\cos x_1 - 1.2x_2 - (1 + 0.8|x_2|)\ \mathrm{sgn}(s)$$

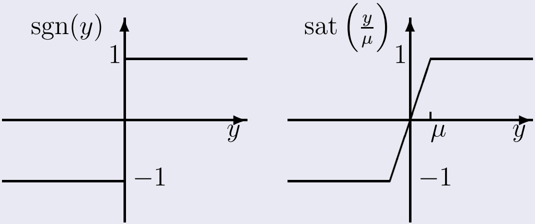


$$b = 0.01, \;\; c = 0.5, \;\; \theta(0) = \dot{\theta}(0) = 0$$

Replace the signum function by a high-slope saturation function

$$u = -\beta(x)\ \mathrm{sat}\left(\frac{s}{\mu}\right)$$

$$\mathrm{sat}(y) = \begin{cases} y, & \text{if } |y| \le 1 \\ \mathrm{sgn}(y), & \text{if } |y| > 1 \end{cases}$$



How can we analyze the system?

$$\text{For } |s| \geq \mu, \quad u = -\beta(x)\,\text{sgn}(s)$$

With $c \geq \mu$

- $\Omega = \left\{|x_1| \leq \frac{c}{a},\ |s| \leq c\right\}$ is positively invariant
- The trajectory reaches the boundary layer $\{|s| \leq \mu\}$in finite time
- The boundary layer is positively invariant

Inside the boundary layer:

$$\dot{x}_1 = -ax_1 + s \qquad \dot{s} = ax_2 + h(x) - g(x)\beta(x)\frac{s}{\mu}$$

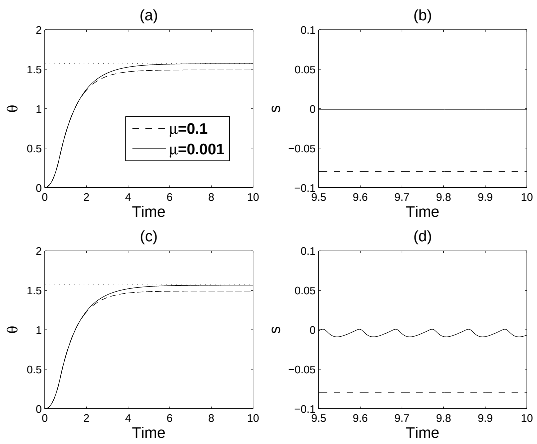
$$x_1\dot{x}_1 \le -ax_1^2 + |x_1|\mu$$

$$x_1\dot{x}_1 \le -(1-\theta_1)ax_1^2, \;\; \forall\; |x_1| \ge \frac{\mu}{\theta_1 a}, \;\; 0 < \theta_1 < 1$$

The trajectories reach the positively invariant set

$$\Omega_\mu = \{|x_1| \le \frac{\mu}{\theta_1 a}, \; |s| \le \mu\}$$

in finite time



(a) & (b) without and (c) & (d) with actuator dynamics

Special case $h(0) = 0$

Inside the boundary layer, the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = h(x) - [g(x)\beta(x)/\mu]\,(ax_1 + x_2)$$

has an equilibrium point at the origin. We can stabilize the origin by choosing μ small enough

Pendulum equation: Stabilize the pendulum at $\theta = \pi$

$$x_1 = \theta - \pi, \quad x_2 = \dot{\theta}, \quad s = x_1 + x_2$$

$$\dot{s} = x_2 + \sin x_1 - bx_2 + cu$$

$$0 \le b \le 0.2, \;\; 0.5 \le c \le 2$$

$$\left| \frac{(1-b)x_2 + \sin x_1}{c} \right| \le 2(|x_1| + |x_2|)$$

$$u = -2(|x_1| + |x_2| + 1)\ \mathrm{sat}(s/\mu)$$

Inside the boundary layer

$$\dot{x}_1 = -x_1 + s, \quad \dot{s} = (1-b)x_2 + \sin x_1 - 2c(|x_1| + |x_2| + 1)s/\mu$$

$$V_1 = \frac{1}{2}x_1^2 + \frac{1}{2}s^2$$

$$\dot{V}_1 \le - \begin{bmatrix} |x_1| \\ |s| \end{bmatrix}^T \begin{bmatrix} 1 & -3/2 \\ -3/2 & (1/\mu \; - \; 1) \end{bmatrix} \begin{bmatrix} |x_1| \\ |s| \end{bmatrix}$$