# Nonlinear Control

## Lecture # 29

## Robust State Feedback Stabilization

# Sliding Mode Control

## Theorem 10.2

Suppose all the assumptions of Theorem 10.1 hold over $\Omega$ with

- $\varrho(0) = 0$
- The origin of $\dot{\eta} = f_a(\eta, \phi(\eta))$ is exponentially stale

Then there exits $\mu^* > 0$ such that for all $0 < \mu < \mu^*$, the origin of the closed-loop system is exponentially stable and $\Omega$ is a subset of its region of attraction. If the assumptions hold globally, the origin will be globally uniformly asymptotically stable

## Proof

By Theorem 10.1, all trajectories starting in $\Omega$ enter $\Omega_\mu$ in finite time. Inside $\Omega_\mu$

$$\dot{\eta} = f_a(\eta, \phi(\eta) + s), \qquad \mu \dot{s} = \beta(x) G(x) s + \mu \Delta(t, x, v)$$

By the converse Lyapunov theorem, there is $V_1(\eta)$ that satisfies

$$c_1 \|\eta\|^2 \le V_1(\eta) \le c_2 \|\eta\|^2$$

$$\frac{\partial V_1}{\partial \eta} f_a(\eta, \phi(\eta)) \le -c_3 \|\eta\|^2$$

$$\left\| \frac{\partial V_1}{\partial \eta} \right\| \le c_4 \|\eta\|$$

in some neighborhood $N_\eta$ of $\eta = 0$

By the smoothness of $f_a$ we have

$$\|f_a(\eta, \phi(\eta) + s) - f_a(\eta, \phi(\eta))\| \leq k_1 \|s\|$$

in some neighborhood $N$ of $(\eta, \xi) = (0, 0)$

Choose $\mu$ small enough that $\Omega_\mu \subset N_\eta \cap N$. Inside $\Omega_\mu$

$$\begin{aligned} s^T \dot{s} &= -\frac{\beta(x)}{\mu} s^T G(x) s + s^T \Delta(t, x, v) \\ &\leq -\frac{\beta \lambda_{min}(G)}{\mu} \|s\|^2 + \lambda_{min}(G) \left[ \varrho + \frac{\kappa_0 \beta \|s\|}{\mu} \right] \|s\| \\ &\leq -\frac{\lambda_0 \beta_0 (1 - \kappa_0)}{\mu} \|s\|^2 + \lambda_{min}(G) \varrho \|s\| \end{aligned}$$

Since $G$ is continuous and $\varrho$ is locally Lipschitz with $\varrho(0) = 0$, we arrive at

$$s^T \dot{s} \leq -\frac{\lambda_0 \beta_0 (1 - \kappa_0)}{\mu} \|s\|^2 + k_2 \|\eta\| \|s\| + k_3 \|s\|^2$$

$$W = V_1(\eta) + \tfrac{1}{2} s^T s$$

$$\begin{aligned} \dot{W} \;\; \leq \;\; & -c_3 \|\eta\|^2 + c_4 k_1 \|\eta\| \|s\| + k_2 \|\eta\| \|s\| \\ & + k_3 \|s\|^2 - \frac{\lambda_0 \beta_0 (1 - \kappa_0)}{\mu} \|s\|^2 \end{aligned}$$

$$\dot{W} \leq - \begin{bmatrix} \|\eta\| \\ \|s\| \end{bmatrix}^T \begin{bmatrix} c_3 & -\frac{c_4 k_1 + k_2}{2} \\ -\frac{c_4 k_1 + k_2}{2} & \frac{\lambda_0 \beta_0 (1-\kappa_0)}{\mu} - k_3 \end{bmatrix} \begin{bmatrix} \|\eta\| \\ \|s\| \end{bmatrix}$$

$$\mu < \frac{4 c_3 \lambda_0 \beta_0 (1 - \kappa_0)}{4 c_3 k_3 + (c_4 k_1 + k_2)^2}$$

The basic idea of the foregoing proof is that, inside the boundary layer, the control

$$v = - \frac{\beta(x) s}{\mu}$$

acts as high-gain feedback for small $\mu$. By choosing $\mu$ small enough, the high-gain feedback stabilizes the origin

## Unmatched Uncertainty

$$\dot{x} = f(x) + B(x)[G(x)u + \delta(t, x, u)] + \delta_1(x)$$

$$\begin{aligned}
\dot{\eta} &= f_a(\eta, \xi) + \delta_a(\eta, \xi) \\
\dot{\xi} &= f_b(\eta, \xi) + G(x)u + \delta(t, x, u) + \delta_b(\eta, \xi)
\end{aligned}$$

$$s = \xi - \phi(\eta)$$

Reduced-order model on the sliding manifold:

$$\dot{\eta} = f_a(\eta, \phi(\eta)) + \delta_a(\eta, \phi(\eta))$$

Design of $\phi$ to stabilize $\eta = 0$ in the presence of $\delta_a$

## Example 10.3

$$\dot{x}_1 = x_2 + \theta_1 x_1 \sin x_2, \qquad \dot{x}_2 = \theta_2 x_2^2 + x_1 + u$$

$$|\theta_1| \le a, \quad |\theta_2| \le b$$

$$x_2 = -kx_1 \quad \Rightarrow \quad \dot{x}_1 = -kx_1 + \theta_1 x_1 \sin x_2$$

$$V_1 = \tfrac{1}{2}x_1^2 \quad \Rightarrow \quad x_1\dot{x}_1 \le -kx_1^2 + ax_1^2$$

$$s = x_2 + kx_1, \quad k > a$$

$$\dot{s} = \theta_2 x_2^2 + x_1 + u + k(x_2 + \theta_1 x_1 \sin x_2)$$

$$u = -x_1 - kx_2 + v \quad \Rightarrow \quad \dot{s} = v + \Delta(x)$$

$$\Delta(x) = \theta_2 x_2^2 + k\theta_1 x_1 \sin x_2$$

$$\Delta(x) = \theta_2 x_2^2 + k\theta_1 x_1 \sin x_2$$
$$|\Delta(x)| \le ak|x_1| + bx_2^2$$

$$\beta(x) = ak|x_1| + bx_2^2 + \beta_0, \qquad \beta_0 > 0$$

By Theorem 10.2,

$$u = -x_1 - kx_2 - \beta(x)\ \mathrm{sat}\left(\frac{s}{\mu}\right)$$

with sufficiently small $\mu$, globally stabilizes the origin

## Example 10.4

$$\dot{x}_1 = x_1 + (1 - \theta_1)x_2, \qquad \dot{x}_2 = \theta_2 x_2^2 + x_1 + u$$

$$|\theta_1| \le a, \quad |\theta_2| \le b$$

$$\dot{x}_1 = x_1 + (1 - \theta_1)x_2$$

Design $x_2$ to robustly stabilize the origin $x_1 = 0$

We must have $|a| < 1$

$$x_2 = -kx_1 \;\Rightarrow\; x_1\dot{x}_1 = x_1^2 - k(1 - \theta_1)x_1^2 \le -[k(1 - a) - 1]x_1^2$$

$$k > \frac{1}{1 - a}$$

$$s = x_2 + kx_1$$

Proceeding as in the previous example, we end up with

$$u = -(1+k)x_1 - kx_2 - \beta(x)\,\text{sat}(s/\mu)$$

$$\beta(x) = bx_2^2 + ak|x_2| + \beta_0, \quad \beta_0 > 0$$

Alternative Approach: Suppose $G(x)$ is a diagonal matrix with positive elements

$$\dot{s} = G(x)v + \Delta(t, x, v)$$

$$\dot{s}_i = g_i(x)v_i + \Delta_i(t, x, v), \quad 1 \le i \le m$$

$$g_i(x) \ge g_0 > 0 \qquad \left| \frac{\Delta_i(t, x, v)}{g_i(x)} \right| \le \varrho(x) + \kappa_0 \max_{1 \le i \le m} |v_i|$$

$$v_i = -\beta(x) \operatorname{sat}(s_i/\mu), \qquad 1 \le i \le m$$

$$V_i = \frac{1}{2} s_i^2$$

$$
\begin{aligned}
\dot{V}_i &= s_i g_i(x) v_i + s_i \Delta_i(t,x,v) \\
&\leq g_i(x)\{s_i v_i + |s_i|[\varrho(x) + \kappa_0 \max_{1\leq i\leq m} |v_i|]\} \\
&\leq g_i(x)[-\beta(x) + \varrho(x) + \kappa_0 \beta(x)]|s_i| \\
&= g_i(x)[-(1-\kappa_0)\beta(x) + \varrho(x)]|s_i| \\
&\leq g_i(x)[-\varrho(x) - (1-\kappa_0)\beta_0 + \varrho(x)]|s_i| \\
&\leq -g_0\beta_0(1-\kappa_0)|s_i|
\end{aligned}
$$

$$
\dot{V}_i \leq -g_0\beta_0(1-\kappa_0)\sqrt{2V_i}
$$

ensures that all trajectories reach the boundary layer

$$
\{|s_i| \leq \mu, \ 1 \leq i \leq m\}
$$

in finite time

Results similar to Theorems 10.1 and 10.2 can be proved with

$$\Omega = \{V_0(\eta) \le c_0\} \times \{|s_i| \le c,\ 1 \le i \le m\}$$

$$c_0 \ge \alpha(c), \quad \alpha(r) = \alpha_2(\alpha_4(r\sqrt{m}))$$

$$\Omega_\mu = \{V_0(\eta) \le \alpha(\mu)\} \times \{|s_i| \le \mu,\ 1 \le i \le m\}$$