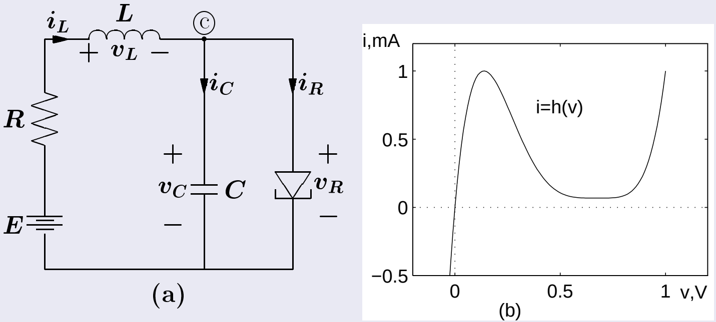
Nonlinear Control

Lecture # 3

Two-Dimensional Systems

Multiple Equilibria

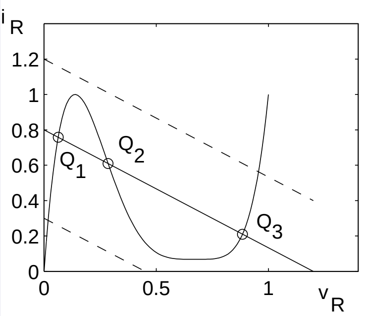
Example 2.2: Tunnel-diode circuit



$$x_1 = v_C, \quad x_2 = i_L$$

$$\begin{aligned}\dot{x}_1 &= 0.5[-h(x_1) + x_2]\\ \dot{x}_2 &= 0.2(-x_1 - 1.5x_2 + 1.2)\end{aligned}$$

$$h(x_1) = 17.76x_1 - 103.79x_1^2 + 229.62x_1^3 - 226.31x_1^4 + 83.72x_1^5$$



$$\begin{aligned}Q_1 &= (0.063, 0.758)\\ Q_2 &= (0.285, 0.61)\\ Q_3 &= (0.884, 0.21)\end{aligned}$$

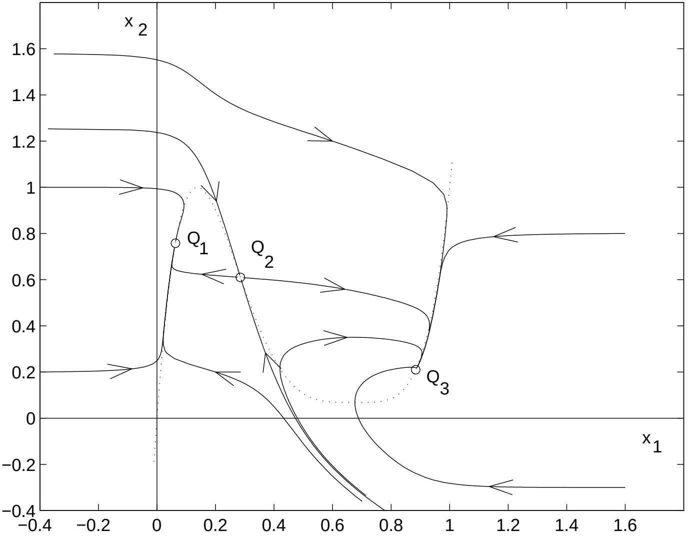
$$\frac{\partial f}{\partial x} = \begin{bmatrix} -0.5h'(x_1) & 0.5 \\ -0.2 & -0.3 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} -3.598 & 0.5 \\ -0.2 & -0.3 \end{bmatrix}, \quad \text{Eigenvalues} : -3.57, \ -0.33$$

$$A_2 = \begin{bmatrix} 1.82 & 0.5 \\ -0.2 & -0.3 \end{bmatrix}, \quad \text{Eigenvalues} : 1.77, \ -0.25$$

$$A_3 = \begin{bmatrix} -1.427 & 0.5 \\ -0.2 & -0.3 \end{bmatrix}, \quad \text{Eigenvalues} : -1.33, \ -0.4$$

Q_1 is a stable node; Q_2 is a saddle; Q_3 is a stable node



x 2
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
Q1
Q2
Q3
x 1

Example 2.3: Pendulum

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\sin x_1 - 0.3x_2$$

Equilibrium points at $(n\pi, 0)$ for $n = 0, \pm 1, \pm 2, \ldots$

$$f(x) = \begin{bmatrix} x_2 \\ -\sin x_1 - 0.3x_2 \end{bmatrix}, \qquad \frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -\cos x_1 & -0.3 \end{bmatrix}$$

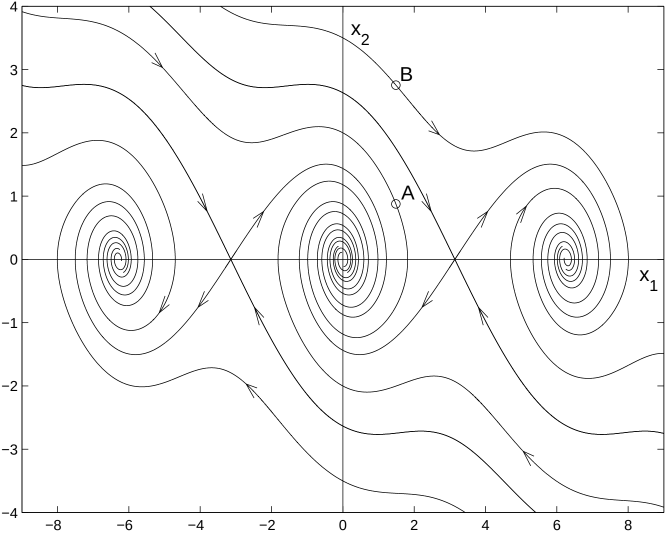
$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -\cos x_1 & -0.3 \end{bmatrix}$$

Linearization at $(0, 0)$ and $(\pi, 0)$:

$$A_1 = \begin{bmatrix} 0 & 1 \\ -1 & -0.3 \end{bmatrix}; \quad \text{Eigenvalues : } -0.15 \pm j0.9887$$

$$A_2 = \begin{bmatrix} 0 & 1 \\ 1 & -0.3 \end{bmatrix}; \quad \text{Eigenvalues : } -1.1612, \; 0.8612$$

$(0, 0)$ is a stable focus and $(\pi, 0)$ is a saddle



4
3
2
1
0
−1
−2
−3
−4
−8
−6
−4
−2
0
2
4
6
8
x2
x1
A
B

Oscillation

A system oscillates when it has a nontrivial periodic solution

$$x(t+T) = x(t), \quad \forall\; t \ge 0$$

Linear (Harmonic) Oscillator:

$$\dot{z} = \begin{bmatrix} 0 & -\beta \\ \beta & 0 \end{bmatrix} z$$

$$z_1(t) = r_0 \cos(\beta t + \theta_0), \qquad z_2(t) = r_0 \sin(\beta t + \theta_0)$$

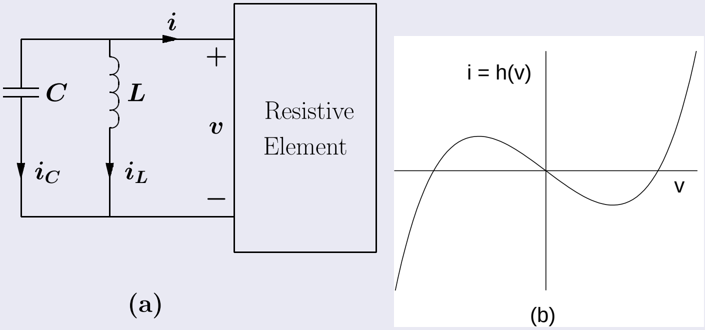
$$r_0 = \sqrt{z_1^2(0) + z_2^2(0)}, \qquad \theta_0 = \tan^{-1}\left[\frac{z_2(0)}{z_1(0)}\right]$$

The linear oscillation is not practical because

- It is not structurally stable. Infinitesimally small perturbations may change the type of the equilibrium point to a stable focus (decaying oscillation) or unstable focus (growing oscillation)
- The amplitude of oscillation depends on the initial conditions
  (The same problems exist with oscillation of nonlinear systems due to a center equilibrium point, e.g., pendulum without friction)

Limit Cycles

Example: Negative Resistance Oscillator



$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 - \varepsilon h'(x_1)x_2
\end{aligned}
$$

There is a unique equilibrium point at the origin

$$
A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} 0 & 1 \\ -1 & -\varepsilon h'(0) \end{bmatrix}
$$

$$
\lambda^2 + \varepsilon h'(0)\lambda + 1 = 0
$$

$h'(0) < 0 \;\Rightarrow\;$ Unstable Focus or Unstable Node

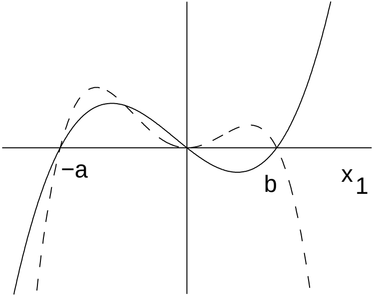
Energy Analysis:

$$E = \tfrac{1}{2}Cv_C^2 + \tfrac{1}{2}Li_L^2$$

$$v_C = x_1 \quad \text{and} \quad i_L = -h(x_1) - \frac{1}{\varepsilon}x_2$$

$$E = \tfrac{1}{2}C\{x_1^2 + [\varepsilon h(x_1) + x_2]^2\}$$

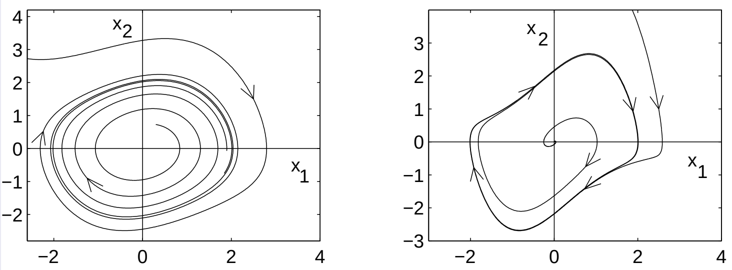
$$\begin{aligned}
\dot{E} &= C\{x_1\dot{x}_1 + [\varepsilon h(x_1) + x_2][\varepsilon h'(x_1)\dot{x}_1 + \dot{x}_2]\} \\
&= C\{x_1x_2 + [\varepsilon h(x_1) + x_2][\varepsilon h'(x_1)x_2 - x_1 - \varepsilon h'(x_1)x_2]\} \\
&= C[x_1x_2 - \varepsilon x_1 h(x_1) - x_1x_2] \\
&= -\varepsilon C x_1 h(x_1)
\end{aligned}$$



$$\dot{E} = -\varepsilon C x_1 h(x_1)$$

Example 2.4: Van der Pol Oscillator

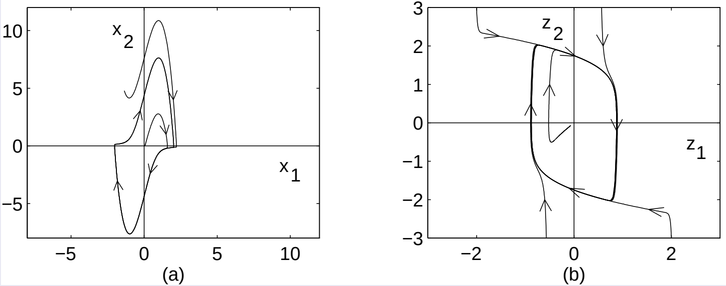
$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \varepsilon(1 - x_1^2)x_2\end{aligned}$$



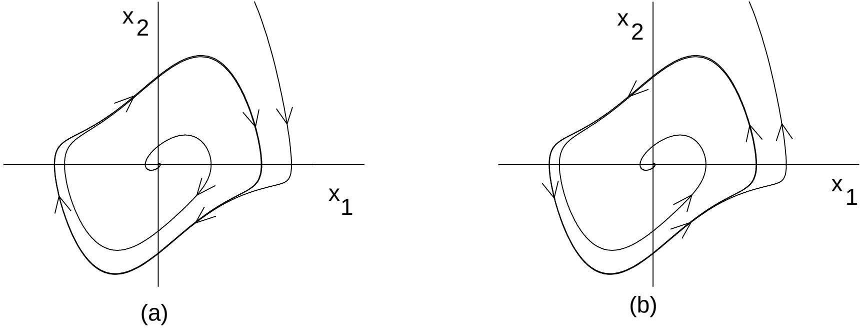
(a) (b)

$\varepsilon = 0.2$ $\varepsilon = 1$

$$
\begin{aligned}
\dot{z}_1 &= \frac{1}{\varepsilon} z_2 \\
\dot{z}_2 &= -\varepsilon\left(z_1 - z_2 + \tfrac{1}{3} z_2^3\right)
\end{aligned}
$$



$\varepsilon = 5$



Stable Limit Cycle

Unstable Limit Cycle