Nonlinear Control

Lecture # 30

Robust State Feedback Stabilization

Lyapunov Redesign

$$\dot{x} = f(x) + G(x)[u + \delta(t, x, u)], \qquad x \in R^n, \; u \in R^m$$

Nominal Model: $\quad \dot{x} = f(x) + G(x)u$

Stabilizing Control: $\quad u = \phi(x)$

$$\frac{\partial V}{\partial x}[f(x) + G(x)\phi(x)] \le -W(x), \;\; \forall \; x \in D, \; W \text{ is p.d.}$$

$$u = \phi(x) + v$$

$$\|\delta(t, x, \phi(x) + v)\| \le \varrho(x) + \kappa_0 \|v\|, \qquad 0 \le \kappa_0 < 1$$

$$\dot{x} = f(x) + G(x)\phi(x) + G(x)[v + \delta(t, x, \phi(x) + v)]$$

$$\dot{V} = \frac{\partial V}{\partial x}(f + G\phi) + \frac{\partial V}{\partial x}G(v + \delta)$$

$$w^T = \frac{\partial V}{\partial x} G$$

$$\dot{V} \leq -W(x) + w^T v + w^T \delta$$

$$w^T v + w^T \delta \leq w^T v + \|w\| \; \|\delta\| \leq w^T v + \|w\| [\varrho(x) + \kappa_0 \|v\|]$$

$$v = -\beta(x) \; \frac{w}{\|w\|} \qquad \left(\frac{w}{\|w\|} = \mathrm{sgn}(w) \text{ for } m = 1 \right)$$

$$\begin{aligned} w^T v + w^T \delta &\leq -\beta \|w\| + \varrho \|w\| + \kappa_0 \beta \|w\| \\ &= -\beta(1 - \kappa_0) \|w\| + \varrho \|w\| \end{aligned}$$

$$\beta(x) \geq \frac{\varrho(x)}{(1 - \kappa_0)} \; \Rightarrow \; w^T v + w^T \delta \leq 0 \; \Rightarrow \; \dot{V} \leq -W(x)$$

Continuous Implementation

$$v = -\beta(x)\ \mathrm{Sat}\left(\frac{\beta(x)w}{\mu}\right) = \begin{cases} -\beta(x)\frac{w}{\|w\|}, & \text{if } \beta(x)\|w\| \geq \mu \\ -\beta^2(x)\frac{w}{\mu}, & \text{if } \beta(x)\|w\| < \mu \end{cases}$$

$$\beta(x)\|w\| \geq \mu \quad \Rightarrow \quad \dot{V} \leq -W(x)$$

For $\beta(x)\|w\| < \mu$

$$\begin{aligned} \dot{V} &\leq -W(x) + w^T\left[-\beta^2 \cdot \frac{w}{\mu} + \delta\right] \\ &\leq -W(x) - \frac{\beta^2}{\mu}\|w\|^2 + \varrho\|w\| + \kappa_0\|w\|\|v\| \\ &= -W(x) - \frac{\beta^2}{\mu}\|w\|^2 + \varrho\|w\| + \frac{\kappa_0\beta^2}{\mu}\|w\|^2 \end{aligned}$$

$$\dot{V} \leq -W(x) + (1-\kappa_0)\left(-\frac{\beta^2}{\mu}\|w\|^2 + \beta\|w\|\right)$$

$$-\frac{y^2}{\mu} + y \leq \frac{\mu}{4}, \quad \text{for } y \geq 0$$

$$\dot{V} \leq -W(x) + \mu\frac{(1-\kappa_0)}{4}$$

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \qquad W(x) \geq \alpha_3(\|x\|)$$

For $0 < \theta < 1$,

$$\begin{aligned}\dot{V} &\leq -(1-\theta)\alpha_3(\|x\|) - \theta\alpha_3(\|x\|) + \mu(1-\kappa_0)/4 \\ &\leq -(1-\theta)\alpha_3(\|x\|), \quad \forall\ \|x\| \geq \alpha_3^{-1}\left(\frac{\mu(1-\kappa_0)}{4\theta}\right) \stackrel{\text{def}}{=} \mu_0\end{aligned}$$

$$B_r \subset D, \quad \mu < \frac{4\theta\alpha_3(\alpha_2^{-1}(\alpha_1(r)))}{(1-\kappa_0)} \ \Rightarrow \ \mu_0 < \alpha_2^{-1}(\alpha_1(r))$$

Theorem 10.3

Under the foregoing assumptions, for any $x(t_0) \in \{V(x) \le \alpha_1(r)\}$, the solution of the closed-loop system satisfies

$$\|x(t)\| \le \max\left\{\beta_1(\|x(t_0)\|, t-t_0),\ b(\mu)\right\}$$

where β_1 is a class $\mathcal{KL}$ function and

$$b(\mu) = \alpha_1^{-1}(\alpha_2(\mu_0)) = \alpha_1^{-1}(\alpha_2(\alpha_3^{-1}(0.25\mu(1-\kappa_0)/\theta)))$$

If all the assumptions hold globally and $\alpha_1 \in \mathcal{K}_\infty$, then the above inequality holds for any initial state $x(t_0)$

Under what conditions will the origin be uniformly asymptotically stable?

$$W(x) \ge \varphi^2(x), \qquad \beta(x) \ge \beta_0 > 0, \qquad \varrho(x) \le \varrho_1 \varphi(x)$$

$$\forall\ x \in B_a = \{\|x\| \le a\}; \quad \varphi(x) \text{ is positive definite}$$

$$\varphi(0) = 0 \quad \Rightarrow \quad \varrho(0) = 0$$

When $\beta(x)\|w\| < \mu$

$$\dot{V} \le -W(x) - \frac{\beta^2(x)(1-\kappa_0)}{\mu}\|w\|^2 + \varrho(x)\|w\|$$

$$-W(x) = -(1-\theta)W(x) - \theta W(x) \le -(1-\theta)W(x) - \theta\varphi^2(x)$$

$$0 < \theta < 1$$

$$
\begin{aligned}
\dot{V} \quad &\leq \quad -(1-\theta)W(x) \\
&\quad - \frac{1}{2}\begin{bmatrix} \varphi(x) \\ \|w\| \end{bmatrix}^T \begin{bmatrix} 2\theta & -\varrho_1 \\ -\varrho_1 & 2\beta_0^2(1-\kappa_0)/\mu \end{bmatrix} \begin{bmatrix} \varphi(x) \\ \|w\| \end{bmatrix}
\end{aligned}
$$

For sufficiently small μ, $\dot{V} \leq -(1-\theta)W(x)$

Theorem 10.4

For sufficiently small μ, the origin is uniformly asymptotically stable. If

$$
c_1\|x\|^2 \leq V(x) \leq c_2\|x\|^2, \quad W(x) \geq c_3\|x\|^2
$$

for $c_i > 0$, then the origin is exponentially stable

Example 10.5 (Pendulum equation)

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\sin(x_1 + \delta_1) - b_0 x_2 + cu$$

$$0 \le b_0 \le 0.2, \quad 0.5 \le c \le 2$$

The system is feedback linearizable

$$\dot{x} = Ax + B[-\sin(x_1 + \delta_1) - b_0 x_2 + cu]$$

Nominal parameters: $\hat{b}_0 = 0$, $\hat{c}$

$$\phi(x) = \left(\frac{1}{\hat{c}}\right)[\sin(x_1 + \delta_1) - k_1 x_1 - k_2 x_2]$$

$K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$ is chosen such that $A - BK$ is Hurwitz

$$u = \phi(x) + v$$

$$\delta = \left(\frac{c - \hat{c}}{\hat{c}^2}\right)[\sin(x_1 + \delta_1) - k_1 x_1 - k_2 x_2] - \frac{b_0}{\hat{c}} x_2 + \left(\frac{c - \hat{c}}{\hat{c}}\right) v$$

$$|\delta| \le \varrho_0 + \varrho_1 |x_1| + \varrho_2 |x_2| + \kappa_0 |v|$$

$$\varrho_0 \ge \left|\frac{(c - \hat{c}) \sin \delta_1}{\hat{c}^2}\right|, \quad \varrho_1 \ge \left|\frac{c - \hat{c}}{\hat{c}^2}\right| (1 + k_1)$$

$$\varrho_2 \ge \frac{b_0}{\hat{c}} + \left|\frac{c - \hat{c}}{\hat{c}^2}\right| k_2, \quad \kappa_0 \ge \left|\frac{c - \hat{c}}{\hat{c}}\right|$$

Assume $\kappa_0 < 1, \ \beta(x) \ge \beta_0 + \dfrac{\varrho_0 + \varrho_1 |x_1| + \varrho_2 |x_2|}{1 - \kappa_0}, \ \beta_0 > 0$

$$V(x) = x^T P x, \quad P(A - BK) + (A - BK)^T P = -I$$

$$w = 2x^T B^T P = 2(p_{12}x_1 + p_{22}x_2)$$

The control

$$u = \phi(x) - \beta(x)\,\mathrm{sat}\left(\beta(x)w/\mu\right)$$

achieves global ultimate boundedness with ultimate bound proportional to $\sqrt{\mu}$. If $\sin\delta_1 = 0$, we take $\varrho_0 = 0$ and the origin of the closed-loop system will be globally exponentially stable

Compare with Example 9.4: There

$$\left|\frac{c-\hat{c}}{\hat{c}}\right| \left[1+\sqrt{k_1^2+k_2^2}\right] \leq \frac{1}{2\sqrt{p_{12}^2+p_{22}^2}}$$

Ultimate bound is proportional to $|\sin(\delta_1)(c-\hat{c})/\hat{c}|$

Here

$$\left|\frac{c-\hat{c}}{\hat{c}}\right| < \kappa_0 < 1$$

Ultimate bound is proportional to $\sqrt{\mu}$

Calculation of the ultimate bound: Let $\delta_1 = \pi/2$

Take $\hat{c} = (2 + 0.5)/2 = 1.25$ to minimze$|(c - \hat{c})/\hat{c}|$

$|c - \hat{c}/\hat{c}| \leq \kappa_0 = 0.6$ Compare with 0.3951 in Example 9.4

$$K = \begin{bmatrix} 1 & 2 \end{bmatrix} \;\Rightarrow\; \lambda(A - BK) = -1, -1 \;\Rightarrow\; w = x_1 + x_2$$

$$\beta(x) = 2.4|x_1| + 2.8|x_2| + 1.5$$

$$u = 0.8(\cos x_1 - x_1 - 2x_2) - \beta(x)\ \mathrm{sat}\left(\beta(x)w/\mu\right)$$

$$\alpha_1(r) = \lambda_{min}(P)r^2,\ \alpha_2(r) = \lambda_{max}(P)r^2,\ \alpha_3(r) = r^2$$

With $\theta = 0.9$,

$$b(\mu) = \alpha_1^{-1}(\alpha_2(\alpha_3^{-1}(0.25\mu(1 - \kappa_0)/\theta))) \approx 0.8\sqrt{\mu}$$

For $x(t)$ to be ultimately in $\{|x_1| \le 0.01,\ |x_2| \le 0.01\}$

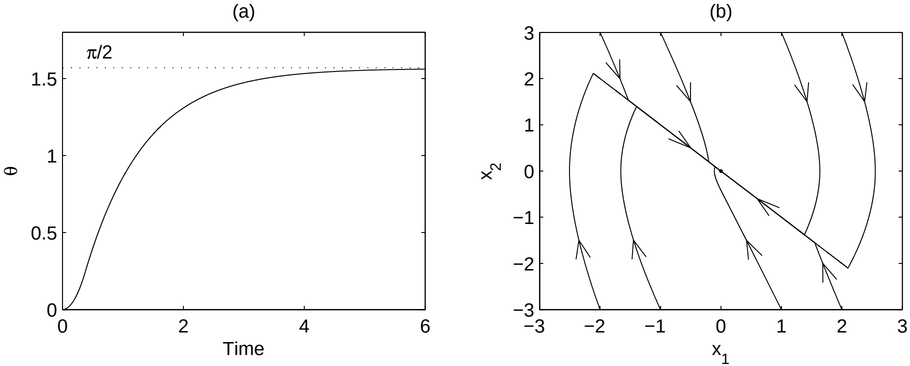
$$\sqrt{\mu} < 0.01/0.8 \ \Rightarrow\ B_b \subset \{|x_1| \le 0.01,\ |x_2| \le 0.01\}$$

Analysis inside B_b yields a less conservative estimate of μ.
Trajectories inside B_b converge to the equilibrium point $(\bar{x}_1 \approx \mu(0.8c - 1)/(2.25c), \bar{x}_2 = 0)$

$$c \in [0.5, 2] \ \Rightarrow\ |\mu(0.8c - 1)/(2.25c)| \le 0.53\mu$$

It is sufficient to choose $\mu < 0.01/0.53$

Simulation: $b_0 = 0.01$, $c = 0.5$, and $\mu = 0.01$



$$\text{when } \beta|w| \geq \mu, \quad w\dot{w} \leq -w^2$$

High-Gain Feedback

The sliding mode and Lyapunov redesign controllers can be replaced by high-gain feedback controllers

Sliding mode control: Replace $\beta(x)\ \mathrm{Sat}\left(\frac{s}{\mu}\right)$ by $\frac{\beta(x)s}{\mu}$

$$V = \frac{1}{2}s^T s$$

$$\begin{aligned}
\dot{V} &= -\frac{\beta}{\mu}s^T G s + s^T \Delta \\
&\le -\frac{\beta}{\mu}\lambda_{min}(G)\|s\|^2 + \lambda_{min}(G)\varrho\|s\| + \lambda_{min}(G)\kappa_0\frac{\beta}{\mu}\|s\|^2 \\
&= \lambda_{min}(G)\left[-\left(\frac{\|s\|}{\mu} - 1\right)\beta(1-\kappa_0)\|s\| - \beta_0(1-\kappa_0)\|s\|\right] \\
&\le -\lambda_0\beta_0(1-\kappa_0)\|s\|, \qquad \text{for } \|s\| \ge \mu
\end{aligned}$$

The trajectories reach $\{\|s\| \leq \mu\}$ in finite time

What is the difference from sliding mode control?

The trajectories reach the boundary layer faster because

$$\mu \dot{s} = -G(x)s + \mu\Delta$$

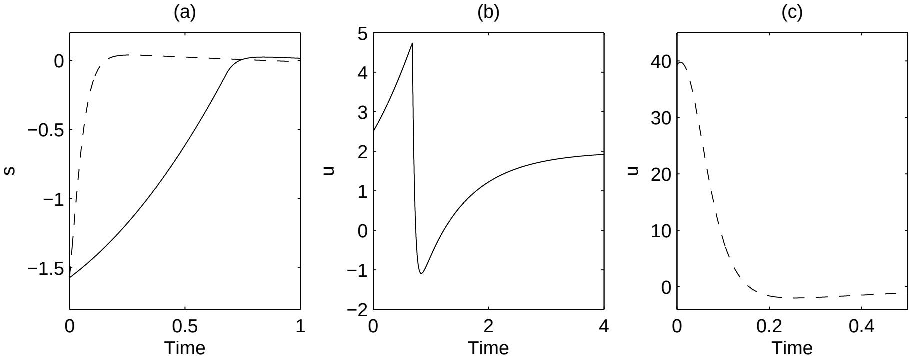
Example 10.6

Recall Example 10.1 where the pendulum is stabilized at $\theta = \pi/2$ and $s = \theta - \pi/2 + \dot{\theta}$

Sliding Mode: $u = -(2.5 + 2|\dot{\theta}|)\,\mathrm{sat}(s/\mu)$

High-gain Feedback: $u = -(2.5 + 2|\dot{\theta}|)(s/\mu)$

Simulation: $b = 0.01,\ c = 0.5,\ \mu = 0.1$



Sliding mode (solid); High-gain (dashed)

Lyapunov redesign: Replace

$$\beta(x)\ \mathrm{Sat}\left(\frac{\beta(x)w}{\mu}\right)$$

by

$$\frac{\beta^2(x)w}{\mu}$$