# Nonlinear Control

## Lecture # 31

## Nonlinear Observers

# Local Observers

$$\dot{x} = f(x, u), \qquad y = h(x)$$

$$\dot{\hat{x}} = f(\hat{x}, u) + H[y - h(\hat{x})]$$

$$\tilde{x} = x - \hat{x}$$

$$\dot{\tilde{x}} = f(x, u) - f(\hat{x}, u) - H[h(x) - h(\hat{x})]$$

We seek a local solution for sufficiently small $\|\tilde{x}(0)\|$

Linearization at $\tilde{x} = 0$:

$$\dot{\tilde{x}} = \left[\frac{\partial f}{\partial x}(x(t), u(t)) - H\frac{\partial h}{\partial x}(x(t))\right]\tilde{x}$$

Steady-state solution:

$$0 = f(x_{ss}, u_{ss}), \qquad 0 = h(x_{ss})$$

Assumption: given $\varepsilon > 0$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$\|x(0) - x_{ss}\| \le \delta_1 \quad \text{and} \quad \|u(t) - u_{ss}\| \le \delta_2 \ \forall\ t \ge\ 0$$

$$\Rightarrow \qquad \|x(t) - x_{ss}\| \le \varepsilon \ \forall\ t \ge\ 0$$

$$A = \frac{\partial f}{\partial x}(x_{ss}, u_{ss}), \qquad C = \frac{\partial h}{\partial x}(x_{ss})$$

Assume that $(A, C)$ is detectable. Design $H$ such that $A - HC$ is Hurwitz

## Lemma 11.1

For sufficiently small $\|\tilde{x}(0)\|$, $\|x(0) - x_{ss}\|$, and $\sup_{t\geq 0} \|u(t) - u_{ss}\|$,

$$\lim_{t\to\infty} \tilde{x}(t) = 0$$

## Proof

$$f(x,u) - f(\hat{x},u) = \int_0^1 \frac{\partial f}{\partial x}(x - \sigma\tilde{x}, u)\ d\sigma\ \tilde{x}$$

$\|f(x,u) - f(\hat{x},u) - A\tilde{x}\| =$

$$\left\| \int_0^1 \left[ \frac{\partial f}{\partial x}(x - \sigma\tilde{x}, u) - \frac{\partial f}{\partial x}(x,u) + \frac{\partial f}{\partial x}(x,u) - \frac{\partial f}{\partial x}(x_{ss}, u_{ss}) \right] d\sigma\ \tilde{x} \right\|$$

$$\leq\ L_1\left(\tfrac{1}{2}\|\tilde{x}\| + \|x - x_{ss}\| + \|u - u_{ss}\|\right)\|\tilde{x}\|$$

$$\|h(x) - h(\hat{x}) - C\tilde{x}\| \le L_2(\tfrac{1}{2}\|\tilde{x}\| + \|x - x_{ss}\|)\|\tilde{x}\|$$

$$\dot{\tilde{x}} = (A - HC)\tilde{x} + \Delta(x, u, \tilde{x})$$

$$\|\Delta(x, u, \tilde{x})\| \le k_1\|\tilde{x}\|^2 + k_2(\varepsilon + \delta_2)\|\tilde{x}\|$$

$$P(A - HC) + (A - HC)^T P = -I$$

$$V = \tilde{x}^T P \tilde{x}$$

$$\dot{V} \le -\|\tilde{x}\|^2 + c_4 k_1\|\tilde{x}\|^3 + c_4 k_2(\varepsilon + \delta_2)\|\tilde{x}\|^2$$

$$\dot{V} \le -\tfrac{1}{3}\|\tilde{x}\|^2, \qquad \text{for } c_4 k_1\|\tilde{x}\| \le \tfrac{1}{3} \quad \text{and} \quad c_4 k_2(\varepsilon + \delta_2) \le \tfrac{1}{3}$$

For sufficiently small $\|\tilde{x}(0)\|$, $\varepsilon$, and $\delta_2$, the estimation error converges to zero as $t$ tends to infinity

# The Extended Kalman Filter

$$\dot{x} = f(x, u), \qquad y = h(x)$$

$$\dot{\hat{x}} = f(\hat{x}, u) + H(t)[y - h(\hat{x})]$$

$$\tilde{x} = x - \hat{x}$$

$$\dot{\tilde{x}} = f(x, u) - f(\hat{x}, u) - H(t)[h(x) - h(\hat{x})]$$

Expand the right-hand side in a Taylor series about $\tilde{x} = 0$ and evaluate the Jacobian matrices along $\hat{x}$

$$\dot{\tilde{x}} = [A(t) - H(t)C(t)]\tilde{x} + \Delta(\tilde{x}, x, u)$$

$$A(t) = \frac{\partial f}{\partial x}(\hat{x}(t), u(t)), \qquad C(t) = \frac{\partial h}{\partial x}(\hat{x}(t))$$

Kalman Filter Gain:

$$H(t) = P(t)C^T(t)R^{-1}$$

$$\dot{P} = AP + PA^T + Q - PC^TR^{-1}CP, \qquad P(t_0) = P_0$$

$P_0$, $Q$ and $R$ are symmetric, positive definite matrices

Assumption 11.1: $P(t)$ exists for all $t \geq t_0$ and satisfies

$$\alpha_1 I \leq P(t) \leq \alpha_2 I, \qquad \alpha_i > 0$$

Remarks:

- Assumption 11.1 cannot be checked offline
- The observer and Riccati equations are solved simultaneously in real time

## Lemma 11.2

There exist positive constants $c$, $k$, and $\lambda$ such that

$$\|\tilde{x}(0)\| \le c \quad \Rightarrow \quad \|\tilde{x}(t)\| \le k e^{-\lambda(t-t_0)}, \quad \forall\ t \ge t_0 \ge 0$$

## Proof

$$\|f(x,u) - f(\hat{x},u) - A(t)\tilde{x}\| =$$

$$\left\| \int_0^1 \left[ \frac{\partial f}{\partial x}(\sigma\tilde{x} + \hat{x}, u) - \frac{\partial f}{\partial x}(\hat{x}, u) \right] d\sigma\ \tilde{x} \right\| \le\ \tfrac{1}{2} L_1 \|\tilde{x}\|^2$$

$$\|h(x) - h(\hat{x}) - C(t)\tilde{x}\| =$$

$$\left\| \int_0^1 \left[ \frac{\partial h}{\partial x}(\sigma\tilde{x} + \hat{x}) - \frac{\partial h}{\partial x}(\hat{x}) \right] d\sigma\ \tilde{x} \right\| \le\ \tfrac{1}{2} L_2 \|\tilde{x}\|^2$$

$$\|C(t)\| = \left\| \frac{\partial h}{\partial x}(x - \tilde{x}) \right\| \leq \left\| \frac{\partial h}{\partial x}(0) \right\| + L_2(\|x\| + \|\tilde{x}\|)$$

$$\|\Delta(\tilde{x}, x, u)\| \leq k_1 \|\tilde{x}\|^2 + k_3 \|\tilde{x}\|^3$$

$$\alpha_1 I \leq P(t) \leq \alpha_2 I \;\Leftrightarrow\; \alpha_3 I \leq P^{-1}(t) \leq \alpha_4 I, \; \alpha_i > 0$$

$$V = \tilde{x}^T P^{-1} \tilde{x}$$

$$\frac{d}{dt} P^{-1} = -P^{-1} \dot{P} P^{-1}$$

$$
\begin{aligned}
\dot{V} &= \tilde{x}^T P^{-1} \dot{\tilde{x}} + \dot{\tilde{x}}^T P^{-1} \tilde{x} + \tilde{x}^T \frac{d}{dt} P^{-1} \tilde{x} \\
&= \tilde{x}^T P^{-1} (A - PC^T R^{-1} C) \tilde{x} \\
&\quad + \tilde{x}^T (A^T - C^T R^{-1} C P) P^{-1} \tilde{x} \\
&\quad - \tilde{x}^T P^{-1} \dot{P} P^{-1} \tilde{x} + 2\tilde{x}^T P^{-1} \Delta \\
&= \tilde{x}^T P^{-1} (AP + PA^T - PC^T R^{-1} C P - \dot{P}) P^{-1} \tilde{x} \\
&\quad - \tilde{x}^T C^T R^{-1} C \tilde{x} + 2\tilde{x}^T P^{-1} \Delta \\
&= -\tilde{x}^T (P^{-1} Q P^{-1} + C^T R^{-1} C) \tilde{x} + 2\tilde{x}^T P^{-1} \Delta
\end{aligned}
$$

$$
\dot{V} \le -c_1 \|\tilde{x}\|^2 + c_2 k_1 \|\tilde{x}\|^3 + c_2 k_2 \|\tilde{x}\|^4
$$

$$
\dot{V} \le -\tfrac{1}{2} c_1 \|\tilde{x}\|^2, \qquad \text{for} \quad \|\tilde{x}\| \le r
$$

## Example 11.1

$$\dot{x} = A_1 x + B_1[0.25x_1^2 x_2 + 0.2\sin 2t], \quad y = C_1 x$$

$$A_1 = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Investigate boundedness of $x(t)$

$$P_1 A_1 + A_1^T P_1 = -I \quad \Rightarrow \quad P_1 = \frac{1}{2}\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$$

$$V(x) = x^T P_1 x$$

$$
\begin{aligned}
\dot{V} &= -x^T x + 2x^T P_1 B_1 [0.25 x_1^2 x_2 + 0.2 \sin 2t] \\
&\leq -\|x\|^2 + 0.5\|P_1 B_1\| x_1^2 \|x\|^2 + 0.4\|P_1 B_1\| \|x\| \\
&= -\|x\|^2 + \frac{x_1^2}{2\sqrt{2}}\|x\|^2 + \frac{0.4}{\sqrt{2}}\|x\| \\
&\leq -0.5\|x\|^2 + \frac{0.4}{\sqrt{2}}\|x\|, \quad \text{for} \;\; x_1^2 \leq \sqrt{2}
\end{aligned}
$$

$$
\frac{\sqrt{2}}{\begin{bmatrix} 1 & 0 \end{bmatrix} P_1^{-1} \begin{bmatrix} 1 & 0 \end{bmatrix}^T} = \sqrt{2}
$$

$$
\Omega = \{V(x) \leq \sqrt{2}\} \subset \{x_1^2 \leq \sqrt{2}\}
$$

Inside $\Omega$

$$\begin{aligned}\dot{V} &\leq -0.5\|x\|^2 + \frac{0.4}{\sqrt{2}}\|x\| \\ &\leq -0.15\|x\|^2, \quad \forall\ \|x\| \geq \frac{0.4}{0.35\sqrt{2}} = 0.8081\end{aligned}$$

$$\lambda_{max}(P_1) = 1.7071 \quad \Rightarrow \quad (0.8081)^2\lambda_{max}(P_1) < \sqrt{2}$$

$$\Rightarrow \quad \{\|x\| \leq 0.8081\} \subset \Omega$$

$\Rightarrow \quad \Omega$ is positively invariant

Design EKF to estimate $x(t)$ for $x(0) \in \Omega$

$$A(t) = \begin{bmatrix} 0 & 1 \\ -1 + 0.5\hat{x}_1(t)\hat{x}_2(t) & -2 + 0.25\hat{x}_1^2(t) \end{bmatrix}$$

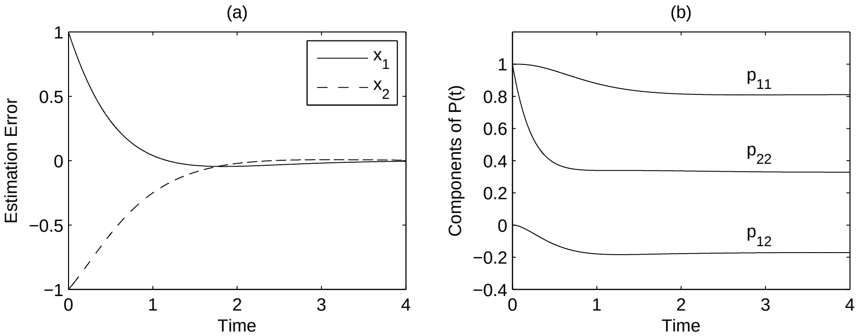
$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$Q = R = P(0) = I$$

$$\dot{P} = AP + PA^T + I - PC^TCP, \qquad P(0) = I$$

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$$

$$
\begin{aligned}
\dot{\hat{x}}_1 &= \hat{x}_2 + p_{11}(y - \hat{x}_1) \\
\dot{\hat{x}}_2 &= -\hat{x}_1 - 2\hat{x}_2 + 0.25\hat{x}_1^2\hat{x}_2 + 0.2\sin 2t + p_{12}(y - \hat{x}_1) \\
\dot{p}_{11} &= 2p_{12} + 1 - p_{11}^2 \\
\dot{p}_{12} &= p_{11}(-1 + 0.5\hat{x}_1\hat{x}_2) + p_{12}(-2 + 0.25\hat{x}_1^2) \\
&\quad + p_{22} - p_{11}p_{12} \\
\dot{p}_{22} &= 2p_{12}(-1 + 0.5\hat{x}_1\hat{x}_2) + 2p_{22}(-2 + 0.25\hat{x}_1^2) \\
&\quad + 1 - p_{12}^2
\end{aligned}
$$



(a)
Estimation Error
Time
x1
x2
1
0.5
0
−0.5
−1
0
1
2
3
4
(b)
Components of P(t)
Time
p11
p22
p12
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
0
1
2
3
4