# Nonlinear Control

## Lecture # 32

## Nonlinear Observers

# Global Observers

Observer Form:

$$\dot{x} = Ax + \psi(u, y), \qquad y = Cx$$

$(A, C)$ observable

$$\dot{\hat{x}} = A\hat{x} + \psi(u, y) + H(y - C\hat{x})$$

$$\tilde{x} = x - \hat{x}$$

$$\dot{\tilde{x}} = (A - HC)\tilde{x}$$

Design $H$ such that $A - HC$ is Hurwitz

$$lim_{t\to\infty}\tilde{x}(t) = 0, \qquad \forall\ \tilde{x}(0)$$

$$\dot{x} = Ax + \psi(u, y) + \phi(x, u), \qquad y = Cx$$

$$\|\phi(x, u) - \phi(z, u)\| \le L\|x - z\|$$

$$\dot{\hat{x}} = A\hat{x} + \psi(u, y) + \phi(\hat{x}, u) + H(y - C\hat{x})$$

$$\dot{\tilde{x}} = (A - HC)\tilde{x} + \phi(x, u) - \phi(\hat{x}, u)$$

$$P(A - HC) + (A - HC)^T P = -I, \qquad V = \tilde{x}^T P \tilde{x}$$

$$\dot{V} = -\tilde{x}^T \tilde{x} + 2\tilde{x}^T P[\phi(x, u) - \phi(\hat{x}, u)] \le -\|\tilde{x}\|^2 + 2L\|P\|\|\tilde{x}\|^2$$

$$L < \frac{1}{2\|P\|} \quad \Rightarrow \quad lim_{t\to\infty}\tilde{x}(t) = 0, \qquad \forall\ \tilde{x}(0)$$

# High-Gain Observers

## Example 11.2

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = \phi(x,u), \qquad y = x_1$$

$$\dot{\hat{x}}_1 = \hat{x}_2 + h_1(y - \hat{x}_1), \qquad \dot{\hat{x}}_2 = \phi_0(\hat{x},u) + h_2(y - \hat{x}_1)$$

$$|\phi_0(z,u) - \phi(x,u)| \le L\|x - z\| + M$$

$$\dot{\tilde{x}} = A_o\tilde{x} + B\delta(x,\tilde{x},u)$$

$$A_o = \begin{bmatrix} -h_1 & 1 \\ -h_2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \delta = \phi(x,u) - \phi_0(\hat{x},u)$$

Design $h_1$ and $h_2$ such that $A_o$ is Hurwitz

Transfer function from $\delta$ to $\tilde{x}$:

$$G_o(s) = \frac{1}{s^2 + h_1 s + h_2} \begin{bmatrix} 1 \\ s + h_1 \end{bmatrix}$$

We can make $\sup_{\omega \in R} \|G_o(j\omega)\|$ arbitrarily small by choosing $h_2 \gg h_1 \gg 1$

$$h_1 = \frac{\alpha_1}{\varepsilon}, \qquad h_2 = \frac{\alpha_2}{\varepsilon^2}, \qquad \varepsilon \ll 1$$

$$G_o(s) = \frac{\varepsilon}{(\varepsilon s)^2 + \alpha_1 \varepsilon s + \alpha_2} \begin{bmatrix} \varepsilon \\ \varepsilon s + \alpha_1 \end{bmatrix}$$

$$\lim_{\varepsilon \to 0} G_o(s) = 0$$

$$\eta_1 = \frac{\tilde{x}_1}{\varepsilon}, \qquad \eta_2 = \tilde{x}_2$$

$$\varepsilon\dot{\eta} = F\eta + \varepsilon B\delta, \qquad \text{where} \quad F = \left[\begin{array}{cc} -\alpha_1 & 1 \\ -\alpha_2 & 0 \end{array}\right]$$

$$PF + F^T P = -I, \qquad V = \eta^T P \eta$$

$$|\delta| \leq L\|\tilde{x}\| + M \leq L\|\eta\| + M$$

$$\begin{aligned} \varepsilon\dot{V} &= -\eta^T\eta + 2\varepsilon\eta^T PB\delta \\ &\leq -\|\eta\|^2 + 2\varepsilon L\|PB\|\|\eta\|^2 + 2\varepsilon M\|PB\|\|\eta\| \end{aligned}$$

$$\varepsilon L\|PB\| \leq \tfrac{1}{4} \quad \Rightarrow \quad \varepsilon \dot{V} \leq -\tfrac{1}{2}\|\eta\|^2 + 2\varepsilon M\|PB\|\|\eta\|$$

$$\|\eta(t)\| \leq \max\left\{ke^{-at/\varepsilon}\|\eta(0)\|,\ \varepsilon c M\right\}, \qquad \forall\ t \geq 0$$

Peaking Phenomenon:

$$x_1(0) \neq \hat{x}_1(0) \quad \Rightarrow \quad \eta_1(0) = O(1/\varepsilon)$$

The solution will contain a term of the form $(1/\varepsilon)e^{-at/\varepsilon}$

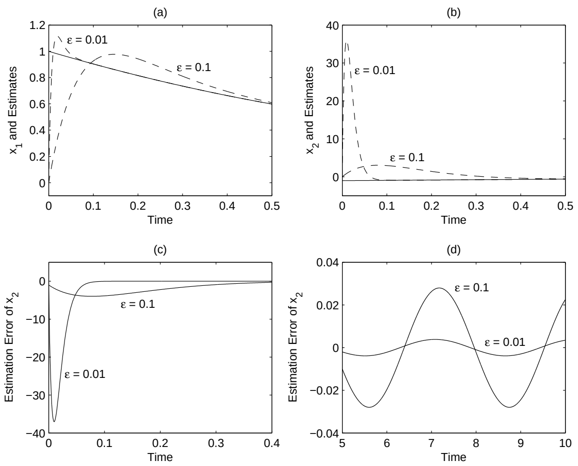
## Example 11.1 (revisited)

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 - 2x_2 + a x_1^2 x_2 + b \sin 2t, \qquad y = x_1$$

$a = 0.25$, $b = 0.2$, and $\Omega = \{1.5x_1^2 + x_1 x_2 + 0.5x_2^2 \le \sqrt{2}\}$ is positively invariant

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + \frac{2}{\varepsilon}(y - \hat{x}_1) \\ \dot{\hat{x}}_2 &= -\hat{x}_1 - 2\hat{x}_2 + \hat{a}\hat{x}_1^2\hat{x}_2 + \hat{b}\sin 2t + \frac{1}{\varepsilon^2}(y - \hat{x}_1) \end{aligned}$$

- Case 1: $\hat{a} = 0.25$ and $\hat{b} = 0.2$ (Figures (a) and (b))
- Case 2: $\hat{a} = \hat{b} = 0$ (Figures (c) and (d))


(a)
1.2
1
0.8
0.6
0.4
0.2
0
ε = 0.01
ε = 0.1
x1 and Estimates
0
0.1
0.2
0.3
0.4
0.5
Time
(b)
40
30
20
10
0
ε = 0.01
ε = 0.1
x2 and Estimates
0
0.1
0.2
0.3
0.4
0.5
Time
(c)
0
−10
−20
−30
−40
ε = 0.1
ε = 0.01
Estimation Error of x2
0
0.1
0.2
0.3
0.4
Time
(d)
0.04
0.02
0
−0.02
−0.04
ε = 0.1
ε = 0.01
Estimation Error of x2
5
6
7
8
9
10
Time

## Measurement Noise

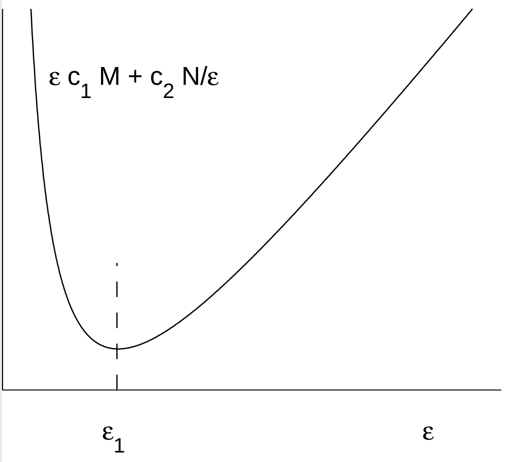
$$y = x_1 + v, \qquad |v(t)| \le N$$

$$\varepsilon\dot{\eta} = F\eta + \varepsilon B\delta - \frac{1}{\varepsilon}Ev, \qquad \text{where} \qquad E = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

$$\varepsilon\dot{V} \le -\tfrac{1}{2}\|\eta\|^2 + 2\varepsilon M\|PB\|\|\eta\| + \frac{2N}{\varepsilon}\|PE\|\|\eta\|$$

Ultimate bound:

$$\|\tilde{x}\| \le c_1 M\varepsilon + \frac{c_2 N}{\varepsilon}$$


ε c1 M + c2 N/ε
ε1
ε

## General case:

$$\begin{aligned}
\dot{w} &= f_0(w, x, u) \\
\dot{x}_i &= x_{i+1} + \psi_i(x_1, \ldots, x_i, u), \qquad \text{for } 1 \le i \le \rho - 1 \\
\dot{x}_\rho &= \phi(w, x, u) \\
y &= x_1
\end{aligned}$$

$$|\psi_i(x_1, \ldots, x_i, u) - \psi_i(z_1, \ldots, z_i, u)| \le L_i \sum_{k=1}^{i} |x_k - z_k|$$

$$\begin{aligned}
\dot{\hat{x}}_i &= \hat{x}_{i+1} + \psi_1(\hat{x}_1, \ldots, \hat{x}_i, u) + \frac{\alpha_i}{\varepsilon^i}(y - \hat{x}_1) \\
\dot{\hat{x}}_\rho &= \phi_0(\hat{x}, u) + \frac{\alpha_\rho}{\varepsilon^\rho}(y - \hat{x}_1)
\end{aligned}$$

$\alpha_1$ to $\alpha_\rho$ are chosen such that the roots of

$$s^\rho + \alpha_1 s^{\rho-1} + \cdots + \alpha_{\rho-1} s + \alpha_\rho = 0$$

have negative real parts

$$|\phi(w, x, u) - \phi_0(x, u)| \leq M$$

$$|\phi_0(x, u) - \phi_0(\hat{x}, u)| \leq L\|x - \hat{x}\|$$

$$|\phi(w, x, u) - \phi_0(\hat{x}, u)| \leq L\|x - \hat{x}\| + M$$

**Lemma 11.3**

For sufficiently small $\varepsilon$

$$|\tilde{x}_i| \leq \max\left\{\frac{b}{\varepsilon^{i-1}}\ e^{-at/\varepsilon},\ \varepsilon^{\rho+1-i} cM\right\}$$

## Proof

$$\eta_1 = \frac{x_1 - \hat{x}_1}{\varepsilon^{\rho-1}}, \quad \dots \quad , \quad \eta_{\rho-1} = \frac{x_{\rho-1} - \hat{x}_{\rho-1}}{\varepsilon}, \quad \eta_\rho = x_\rho - \hat{x}_\rho$$

$$\varepsilon\dot{\eta} = F\eta + \varepsilon\delta(w, x, \tilde{x}, u)$$

$$\delta = \mathrm{col}(\delta_1, \delta_2, \cdots, \delta_\rho), \quad \delta_\rho = \phi(w, x, u) - \phi_0(\hat{x}, u)$$

$$|\delta_i| \le L_i \sum_{k=1}^{i} \varepsilon^{i-k}|\eta_k|, \quad 1 \le i \le \rho - 1$$

$$F = \begin{bmatrix} -\alpha_1 & 1 & 0 & \cdots & 0 \\ -\alpha_2 & 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ -\alpha_{\rho-1} & & & 0 & 1 \\ -\alpha_\rho & 0 & \cdots & \cdots & 0 \end{bmatrix}$$

For $\varepsilon \leq \varepsilon^*$

$$\|\delta\| \leq L_\delta \|\eta\| + M$$

$$V = \eta^T P \eta, \quad PF + F^T P = -I$$

$$\varepsilon \dot{V} = -\eta^T \eta + 2\varepsilon \eta^T P \delta$$

$$\varepsilon \dot{V} \leq -\|\eta\|^2 + 2\varepsilon \|P\| L_\delta \|\eta\|^2 + 2\varepsilon \|P\| M \|\eta\|$$

For $\epsilon \|P\| L_\delta \leq \frac{1}{4}$,

$\varepsilon \dot{V} \leq -\frac{1}{2}\|\eta\|^2 + 2\varepsilon \|P\| M \|\eta\| \leq -\frac{1}{4}\|\eta\|^2, \ \forall \ \|\eta\| \geq 8\varepsilon \|P\| M$

By Theorem 4.5, $\quad \|\eta(t)\| \leq \max\left\{ke^{-at/\varepsilon}\|\eta(0)\|, \ \varepsilon c M\right\}$

$$|\tilde{x}_i| \leq \varepsilon^{\rho - i} |\eta_i|$$