# Nonlinear Control

## Lecture # 33

## Output Feedback Stabilization

# Linearization

$$\dot{x} = f(x,u), \quad y = h(x), \quad f(0,0) = 0, \quad h(0) = 0$$

$$\dot{x} = Ax + Bu \qquad y = Cx$$

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0,u=0}, \quad B = \left.\frac{\partial f}{\partial u}\right|_{x=0,u=0}, \quad C = \left.\frac{\partial h}{\partial x}\right|_{x=0}$$

Design a linear output feedback controller:

$$\dot{z} = Fz + Gy, \qquad u = Lz + My$$

such that the closed-loop matrix

$$\begin{bmatrix} A + BMC & BL \\ GC & F \end{bmatrix} \qquad \text{is Hurwitz}$$

## Examples

- Static output feedback controller

$$u = My$$

where $A + BMC$ is Hurwitz

- Observer-based controller

$$\dot{\hat{x}} = A\hat{x} + Bu + H(y - C\hat{x}), \qquad u = -K\hat{x}$$

The closed-loop matrix Hurwitz if $A - BK$ and $A - HC$ are Hurwitz

Closed-loop system:

$$\dot{x} = f(x, Lz + Mh(x)), \qquad \dot{z} = Fz + Gh(x)$$

Linearization at the origin $(x = 0,\ z = 0)$ results in the Hurwitz matrix

$$\begin{bmatrix} A + BMC & BL \\ GC & F \end{bmatrix}$$

By Theorem 3.2, the origin is exponentially stable

# Passivity-Based Control

In Section 9.6 we saw that if the system

$$\dot{x} = f(x, u), \qquad y = h(x)$$

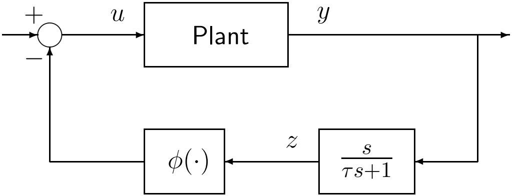
is passive (with a positive definite storage function) and zero-state observable, it can be stabilized by

$$u = -\phi(y), \quad \phi(0) = 0, \quad y^T\phi(y) > 0, \quad \forall\ y \neq 0$$

Suppose the system

$$\dot{x} = f(x, u), \qquad \dot{y} = \frac{\partial h}{\partial x} f(x, u) \stackrel{\text{def}}{=} \tilde{h}(x, u)$$

is passive (with a positive definite storage function $V(x)$) and zero state observable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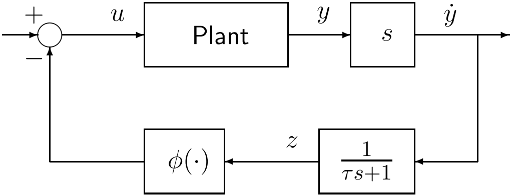
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$$\frac{s}{\tau s + 1}$$

$$\tau \dot{w} = -w + y, \qquad z = (-w + y)/\tau$$

MIMO systems

$$\tau_i \dot{w}_i = -w_i + y_i, \quad z_i = (-w_i + y_i)/\tau_i, \quad \text{for } 1 \le i \le m$$

Note that

$$\tau_i \dot{z}_i = -z_i + \dot{y}_i$$

## Lemma 12.1

Consider the system

$$\dot{x} = f(x,u), \qquad y = h(x)$$

and the output feedback controller

$$u_i = -\phi_i(z_i), \quad \tau_i \dot{w}_i = -w_i + y_i, \quad z_i = (-w_i + y_i)/\tau_i$$

$$\tau_i > 0,\ \ \phi_i(0) = 0,\ \ z_i\phi_i(z_i) > 0 \ \forall\ z_i \neq 0$$

Suppose the auxiliary system

$$\dot{x} = f(x,u), \qquad \dot{y} = \tilde{h}(x,u)$$

is

- passive with a positive definite storage function $V(x)$

$$u^T \dot{y} \geq \dot{V} = \frac{\partial V}{\partial x} f(x, u), \qquad \forall \ (x, u)$$

- zero-state observable

$$\text{with } u = 0, \ \dot{y}(t) \equiv 0 \ \Rightarrow \ x(t) \equiv 0$$

Then the origin of the closed-loop system is asymptotically stable. It is globally asymptotically stable if $V(x)$ is radially unbounded and $\int_0^{z_i} \phi_i(\sigma) \ d\sigma \rightarrow \infty$ as $|z_i| \rightarrow \infty$

## Proof

$$W(x,z) = V(x) + \sum_{i=1}^{m} \tau_i \int_0^{z_i} \phi_i(\sigma)\ d\sigma$$

$$\dot{W} = \dot{V} + \sum_{i=1}^{m} \tau_i \phi_i(z_i) \dot{z}_i \le u^T \dot{y} - \sum_{i=1}^{m} z_i \phi_i(z_i) - u^T \dot{y}$$

$$\dot{W} \le -\sum_{i=1}^{m} z_i \phi_i(z_i)$$

$$\dot{W} \equiv 0 \quad \Rightarrow \quad z(t) \equiv 0 \quad \Rightarrow \quad u(t) \equiv 0 \quad \text{and} \quad \dot{y}(t) \equiv 0$$

Apply the invariance principle

## Example 12.2 ($m$-link Robot Manipulator)

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + D\dot{q} + g(q) = u$$

$$M = M^T > 0, \ (\dot{M} - 2C)^T = -(\dot{M} - 2C), \ D = D^T \geq 0$$

Stabilize the system at $q = q_r$, $e = q - q_r$, $\dot{e} = \dot{q}$

$$M(q)\ddot{e} + C(q,\dot{q})\dot{e} + D\dot{e} + g(q) = u$$

$$u = g(q) - K_p e + v, \quad [K_p = K_p > 0]$$

$$M(q)\ddot{e} + C(q,\dot{q})\dot{e} + D\dot{e} + K_p e = v, \qquad y = e$$

$$V = \tfrac{1}{2}\dot{e}^T M(q)\dot{e} + \tfrac{1}{2}e^T K_p e$$

$$V = \frac{1}{2}\dot{e}^T M(q)\dot{e} + \frac{1}{2}e^T K_p e$$

$$\dot{V} = \frac{1}{2}\dot{e}^T(\dot{M} - 2C)\dot{e} - \dot{e}^T D\dot{e} - \dot{e}^T K_p e + \dot{e}^T v + e^T K_p \dot{e} \ \leq \ \dot{e}^T v$$

Is it zero-state observable? Set $v = 0$

$$\dot{e}(t) \equiv 0 \ \Rightarrow \ \ddot{e}(t) \equiv 0 \ \Rightarrow \ K_p e(t) \equiv 0 \ \Rightarrow \ e(t) \equiv 0$$

$$\tau_i \dot{w}_i = -w_i + e_i, \quad z_i = (-a_i w_i + e_i)/\tau_i, \quad \text{for } 1 \leq i \leq m$$

$$u = g(q) - K_p(q - q_r) - K_d z$$

$K_d$ is positive diagonal matrix. Compare with state feedback

$$u = g(q) - K_p(q - q_r) - K_d \dot{q}$$

# Observer-Based Control

$$\dot{x} = f(x, u), \qquad y = h(x)$$

State Feedback Controller: Design a locally Lipschitz $u = \gamma(x)$ to stabilize the origin of

$$\dot{x} = f(x, \gamma(x))$$

Observer:

$$\dot{\hat{x}} = f(\hat{x}, u) + H[y - h(\hat{x})]$$

$$\tilde{x} = x - \hat{x}$$

$$\dot{\tilde{x}} = f(x, u) - f(\hat{x}, u) - H[h(x) - h(\hat{x})] \stackrel{\text{def}}{=} g(x, \tilde{x})$$

$$g(x, 0) = 0$$

Design $H$ such that $\dot{\tilde{x}} = g(x, \tilde{x})$ has an exponentially stable equilibrium point at $\tilde{x} = 0$ and there is Lyapunov function $V_1(\tilde{x})$ such that

$$c_1\|\tilde{x}\|^2 \le V_1 \le c_2\|\tilde{x}\|^2, \ \frac{\partial V_1}{\partial \tilde{x}} g \le -c_3\|\tilde{x}\|^2, \ \left\|\frac{\partial V_1}{\partial \tilde{x}}\right\| \le c_4\|\tilde{x}\|$$

$$u = \gamma(\hat{x})$$

Closed-loop system:

$$\dot{x} = f(x, \gamma(x - \tilde{x})), \qquad \dot{\tilde{x}} = g(x, \tilde{x}) \qquad (\star)$$

## Theorem 12.1

- If the origin of $\dot{x} = f(x, \gamma(x))$ is asymptotically stable, so is the origin of $(\star)$
- If the origin of $\dot{x} = f(x, \gamma(x))$ is exponentially stable, so is the origin of $(\star)$
- If the assumptions hold globally and the system $\dot{x} = f(x, \gamma(x - \tilde{x}))$, with input $\tilde{x}$, is ISS, then the origin of $(\star)$ is globally asymptotically stable

## Proof

If the origin of $\dot{x} = f(x, \gamma(x))$ is asymptotically stable, then by the (converse Lyapunov) Theorem 3.9

$$\frac{\partial V_0}{\partial x} f(x, \gamma(x)) \leq -W_0(x)$$

$$\left\| \frac{\partial V_0}{\partial x} [f(x, \gamma(x)) - f(x, \gamma(x - \tilde{x}))] \right\| \leq L \|\tilde{x}\|$$

$$V(x, \tilde{x}) = bV_0(x) + \sqrt{V_1(\tilde{x})}, \quad b > 0$$

$$\dot{V} = b \frac{\partial V_0}{\partial x} f(x, \gamma(x - \tilde{x})) + \frac{1}{2\sqrt{V_1}} \frac{\partial V_1}{\partial \tilde{x}} g(x, \tilde{x})$$

$$f(x, \gamma(x - \tilde{x})) = f(x, \gamma(x)) + [f(x, \gamma(x - \tilde{x})) - f(x, \gamma(x))]$$

$$\dot{V} \leq -bW_0(x) + bL\|\tilde{x}\| - \frac{c_3\|\tilde{x}\|^2}{2\sqrt{V_1}}$$

$$V_1 \leq c_2\|\tilde{x}\|^2 \;\Rightarrow\; \frac{-1}{\sqrt{V_1}} \leq \frac{-1}{\sqrt{c_2}\;\|\tilde{x}\|}$$

$$\dot{V} \leq -bW_0(x) + bL\|\tilde{x}\| - \frac{c_3}{2\sqrt{c_2}}\|\tilde{x}\|$$

$b < c_3/(2L\sqrt{c_2})$ ensure that $\dot{V}$ is negative definite

If the origin of $\dot{x} = f(x, \gamma(x))$ is exponentially stable, then by (the converse Lyapunov) Theorem 3.8

$$a_1\|x\|^2 \leq V_0(x) \leq a_2\|x\|^2$$

$$\frac{\partial V_0}{\partial x} f(x, \gamma(x)) \leq -a_3\|x\|^2, \quad \left\|\frac{\partial V_0}{\partial x}\right\| \leq a_4\|x\|$$

$$V(x,\tilde{x}) = bV_0(x) + V_1(\tilde{x}), \quad b > 0$$

$$\begin{aligned}
\dot{V} &= b\frac{\partial V_0}{\partial x} f(x, \gamma(x - \tilde{x})) + \frac{\partial V_1}{\partial \tilde{x}} g(x, \tilde{x}) \\
&\le -ba_3\|x\|^2 + ba_4 L_1 \|x\|\|\tilde{x}\| - c_3\|\tilde{x}\|^2 \\
&= -\begin{bmatrix} \|x\| \\ \|\tilde{x}\| \end{bmatrix}^T \underbrace{\begin{bmatrix} ba_3 & -ba_4 L_1/2 \\ -ba_4 L_1/2 & c_3 \end{bmatrix}}_{Q} \begin{bmatrix} \|x\| \\ \|\tilde{x}\| \end{bmatrix}
\end{aligned}$$

$b < 4a_3c_3/(a_4L_1)^2$ ensures that $Q$ is positive definite

The proof of the third bullet follows from Lemma 4.6