Nonlinear Control

Lecture # 34

Output Feedback Stabilization

High-Gain Observers

Example 12.3

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = \phi(x, u), \quad y = x_1$$

State feedback control: $u = \gamma(x)$ stabilizes the origin of

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = \phi(x, \gamma(x))$$

High-gain observer

$$\dot{\hat{x}}_1 = \hat{x}_2 + (\alpha_1/\varepsilon)(y - \hat{x}_1), \quad \dot{\hat{x}}_2 = \phi_0(\hat{x}, u) + (\alpha_2/\varepsilon^2)(y - \hat{x}_1)$$

ϕ_0 is a nominal model of ϕ, $\alpha_i > 0$, $0 < \varepsilon \ll 1$

$$|\tilde{x}_1| \le \max\left\{be^{-at/\varepsilon}, \ \varepsilon^2 cM\right\}, \quad |\tilde{x}_2| \le \max\left\{\frac{b}{\varepsilon}e^{-at/\varepsilon}, \ \varepsilon cM\right\}$$

The bound on $\tilde{x}_2$ demonstrates the peaking phenomenon, which might destabilize the closed-loop system

Example:

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = x_2^3 + u, \qquad y = x_1$$

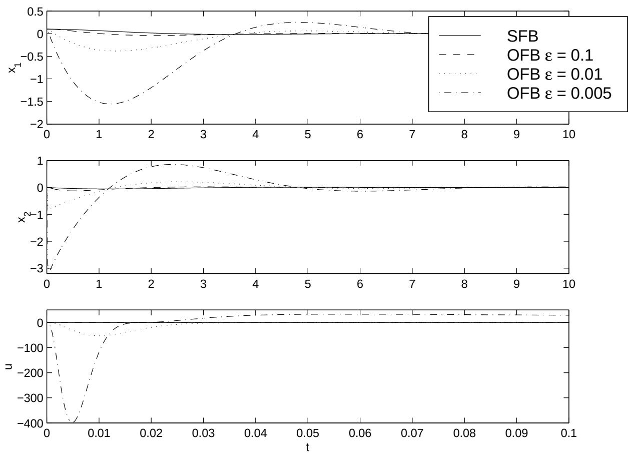
State feedback control:

$$u = -x_2^3 - x_1 - x_2$$

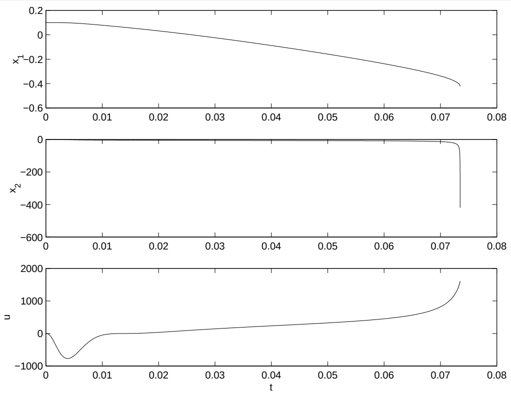
Output feedback control:

$$u = -\hat{x}_2^3 - \hat{x}_1 - \hat{x}_2$$

$$\dot{\hat{x}}_1 = \hat{x}_2 + (2/\varepsilon)(y - \hat{x}_1), \quad \dot{\hat{x}}_2 = (1/\varepsilon^2)(y - \hat{x}_1)$$


SFB
OFB ε = 0.1
OFB ε = 0.01
OFB ε = 0.005
x1
x2
u
t

$\varepsilon = 0.004$



Closed-loop system under state feedback:

$$\dot{x} = Ax, \qquad A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

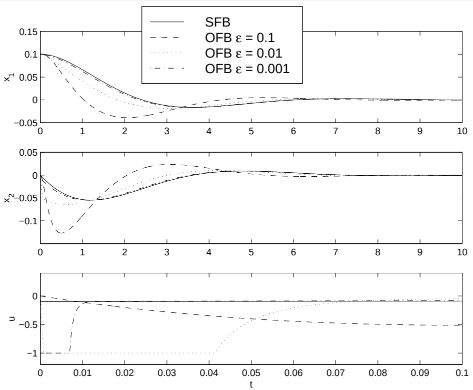
$$PA + A^T P = -I \qquad \Rightarrow \qquad P = \begin{bmatrix} 1.5 & 0.5 \\ 0.5 & 1 \end{bmatrix}$$

Suppose $x(0)$ belongs to the positively invariant set $\Omega = \{V(x) \leq 0.3\}$

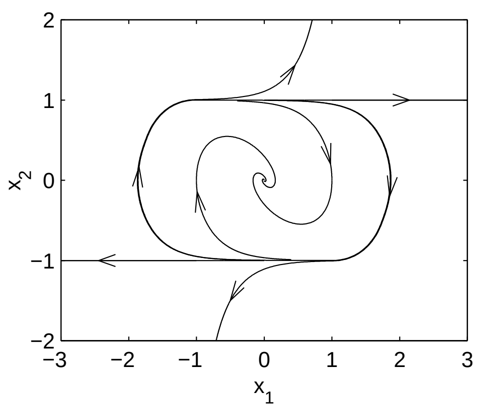
$$|u| \leq |x_2|^3 + |x_1 + x_2| \leq 0.816, \quad \forall\, x \in \Omega$$

Saturate u at ± 1

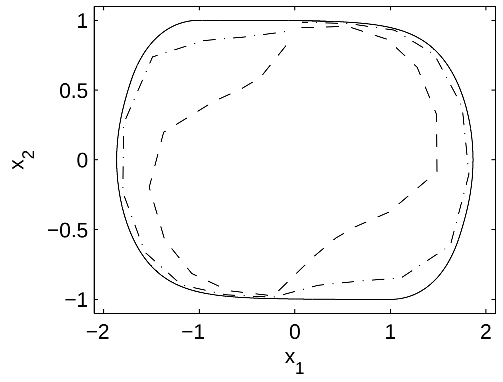
$$u = \mathrm{sat}(-\hat{x}_2^3 - \hat{x}_1 - \hat{x}_2)$$



Region of attraction under state feedback:



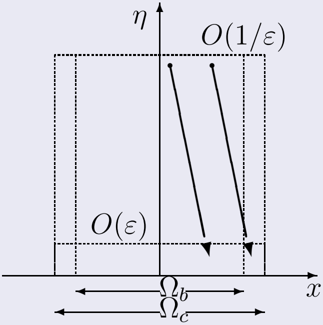
Region of attraction under output feedback:



$\varepsilon = 0.08$ (dashed) and $\varepsilon = 0.01$ (dash-dot)

Analysis of the closed-loop system:

$$\dot{x}_1 = x_2 \qquad \dot{x}_2 = \phi(x, \gamma(x - \tilde{x}))$$
$$\varepsilon\dot{\eta}_1 = -\alpha_1\eta_1 + \eta_2 \qquad \varepsilon\dot{\eta}_2 = -\alpha_2\eta_1 + \varepsilon\delta(x, \tilde{x})$$



General case

$$
\begin{aligned}
\dot{w} &= \psi(w, x, u) \\
\dot{x}_i &= x_{i+1} + \psi_i(x_1, \ldots, x_i, u), \qquad 1 \le i \le \rho - 1 \\
\dot{x}_\rho &= \phi(w, x, u) \\
y &= x_1 \\
z &= q(w, x)
\end{aligned}
$$

$$\phi(0, 0, 0) = 0, \quad \psi(0, 0, 0) = 0, \quad q(0, 0) = 0$$

ψ_i satisfies a global Lipschitz condition. The normal form and models of mechanical and electromechanical systems take this form with

$$\psi_1 = \cdots = \psi_\rho = 0$$

Why the extra measurement z?

In many problems, we can measure some state variables in addition to y

Magnetic levitation system

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -bx_2 + 1 - \frac{4cx_3^2}{(1+x_1)^2}\\ \dot{x}_3 &= \frac{1}{T(x_1)}\left[-x_3 + u + \frac{\beta x_2 x_3}{(1+x_1)^2}\right]\end{aligned}$$

Typical measurements are the ball position x_1 and the current x_3

Stabilizing state feedback controller:

$$\dot{\vartheta} = \Gamma(\vartheta, x, z), \qquad u = \gamma(\vartheta, x, z)$$

γ and Γ are globally bounded functions of x

Closed-loop system

$$\dot{\mathcal{X}} = f(\mathcal{X}), \quad \mathcal{X} = \mathrm{col}(w, x, \vartheta)$$

Output feedback controller

$$\dot{\vartheta} = \Gamma(\vartheta, \hat{x}, z), \quad u = \gamma(\vartheta, \hat{x}, z)$$

Observer

$$\begin{aligned}\dot{\hat{x}}_i &= \hat{x}_{i+1} + \psi_i(\hat{x}_1, \ldots, \hat{x}_i, u) + \frac{\alpha_i}{\varepsilon^i}(y - \hat{x}_1), \\ &\qquad\qquad\qquad\qquad\qquad\qquad 1 \le i \le \rho - 1 \\ \dot{\hat{x}}_\rho &= \phi_0(z, \hat{x}, u) + \frac{\alpha_\rho}{\varepsilon^\rho}(y - \hat{x}_1)\end{aligned}$$

$\varepsilon > 0$ and α_1 to α_ρ are chosen such that the roots of

$$s^\rho + \alpha_1 s^{\rho-1} + \cdots + \alpha_{\rho-1} s + \alpha_\rho = 0$$

have negative real parts

Separation Principle

Theorem 12.2

Suppose the origin of $\dot{\mathcal{X}} = f(\mathcal{X})$ is asymptotically stable and $\mathcal{R}$ is its region of attraction. Let $\mathcal{S}$ be any compact set in the interior of $\mathcal{R}$ and $\mathcal{Q}$ be any compact subset of R^ρ. Then, given any $\mu > 0$ there exist $\varepsilon^* > 0$ and $T^* > 0$, dependent on μ, such that for every $0 < \varepsilon \leq \varepsilon^*$, the solutions $(\mathcal{X}(t), \hat{x}(t))$ of the closed-loop system, starting in $\mathcal{S} \times \mathcal{Q}$, are bounded for all $t \geq 0$ and satisfy

$$\|\mathcal{X}(t)\| \leq \mu \quad \text{and} \quad \|\hat{x}(t)\| \leq \mu, \quad \forall\, t \geq T^*$$

$$\|\mathcal{X}(t) - \mathcal{X}_r(t)\| \leq \mu, \quad \forall\, t \geq 0$$

where $\mathcal{X}_r$ is the solution of $\dot{\mathcal{X}} = f(\mathcal{X})$, starting at $\mathcal{X}(0)$

If the origin of $\dot{\mathcal{X}} = f(\mathcal{X})$ is exponentially stable, then the origin of the closed-loop system is exponentially stable and $\mathcal{S} \times \mathcal{Q}$ is a subset of its region of attraction