Nonlinear Control

Lecture # 35

Output Feedback Stabilization

Robust Stabilization of Minimum Phase Systems

Relative Degree One

$$\dot{\eta} = f_0(\eta, y), \qquad \dot{y} = a(\eta, y) + b(\eta, y)u + \delta(t, \eta, y, u)$$

$$f_0(0,0) = 0, \quad a(0,0) = 0, \quad b(\eta, y) \geq b_0 > 0$$

The origin of $\dot{\eta} = f_0(\eta, 0)$ is asymptotically stable

$$\alpha_1(\|\eta\|) \leq V(\eta) \leq \alpha_2(\|\eta\|)$$

$$\frac{\partial V}{\partial \eta} f_0(\eta, y) \leq -\alpha_3(\|\eta\|), \qquad \forall \ \|\eta\| \geq \alpha_4(|y|)$$

Sliding Mode Control: Sliding surface $y = 0$

$$u = \psi(y) + v$$

$$\left|\frac{a(\eta,y)+b(\eta,y)\psi(y)+\delta(t,\eta,y,\psi(y)+v)}{b(\eta,y)}\right| \le \varrho(y)+\kappa_0|v|$$

$$0 \le \kappa_0 < 1$$

$$\beta(y) \ge \frac{\varrho(y)}{1-\kappa_0} + \beta_0$$

$$v = -\beta(y)\ \mathrm{sat}\left(\frac{y}{\mu}\right)$$

$$u = \psi(y) - \beta(y)\ \mathrm{sat}\left(\frac{y}{\mu}\right)$$

All the assumptions hold in a domain D

Theorem 12.3

Define the class $\mathcal{K}$ function α by $\alpha(r) = \alpha_2(\alpha_4(r))$ and suppose μ, $c > \mu$, and $c_0 \geq \alpha(c)$ are chosen such that the set

$$\Omega = \{V(\eta) \leq c_0\} \times \{|y| \leq c\}, \quad \text{with } c_0 \geq \alpha(c)$$

is compact and contained in D. Then, Ω is positively invariant and for any initial state in Ω, the state is bounded for all $t \geq 0$ and reaches the positively invariant set

$$\Omega_\mu = \{V(\eta) \leq \alpha(\mu)\} \times \{|y| \leq \mu\}$$

in finite time. Moreover, if the assumptions hold globally and $V(\eta)$ is radially unbounded, the foregoing conclusion holds for any initial state

Theorem 12.4

Suppose $\varrho(0) = 0$ and the origin of $\dot{\eta} = f_0(\eta, 0)$ is exponentially stable. Then, there exists $\mu^* > 0$ such that for all $0 < \mu < \mu^*$, the origin of the closed-loop system is exponentially stable and Ω is a subset of its region of attraction. Moreover, if the assumptions hold globally and $V(\eta)$ is radially unbounded, the origin will be globally uniformly asymptotically stable

Relative Degree Higher Than One

$$\begin{aligned}
\dot{\eta} &= f_0(\eta, \xi) \\
\dot{\xi}_i &= \xi_{i+1}, \quad \text{for } 1 \leq i \leq \rho - 1 \\
\dot{\xi}_\rho &= a(\eta, \xi) + b(\eta, \xi)u + \delta(t, \eta, \xi, u) \\
y &= \xi_1
\end{aligned}$$

$$f_0(0,0) = 0, \quad a(0,0) = 0, \quad b(\eta, \xi) \geq b_0 > 0$$

The origin of $\dot{\eta} = f_0(\eta, 0)$ is asymptotically stable

Partial State Feedback: Assume ξ is available for feedback

$$s = k_1\xi_1 + k_2\xi_2 + \cdots + k_{\rho-1}\xi_{\rho-1} + \xi_\rho$$

With s as the output, the system has relative degree one and the normal form is given by

$$\dot{z} = \bar{f}_0(z, s), \qquad \dot{s} = \bar{a}(z, s) + \bar{b}(z, s)u + \bar{\delta}(t, z, s, u)$$

$$z = \text{col}\left(\eta, \quad \xi_1, \quad \dots \quad \xi_{\rho-2}, \quad \xi_{\rho-1}\right)$$

Zero Dynamics ($s = 0$):

$$\dot{z} = \bar{f}_0(z, 0)$$

$$\dot{z} = \bar{f}_0(z, 0)$$

$$\Leftrightarrow \quad \dot{\eta} = \left. f_0(\eta, \xi) \right|_{\xi_\rho = -\sum_{i=1}^{\rho-1} k_i \xi_i}, \quad \dot{\zeta} = F\zeta$$

$$\zeta = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_{\rho-1} \end{bmatrix}, \qquad F = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & & 0 & 1 \\ -k_1 & -k_2 & \cdots & -k_{\rho-2} & -k_{\rho-1} \end{bmatrix}$$

When $\rho = n$, the zero dynamics are $\dot{\zeta} = F\zeta$

k_1 to $k_{\rho-1}$ are chosen such that the polynomial

$$\lambda^{\rho-1} + k_{\rho-1}\lambda^{\rho-2} + \cdots + k_2\lambda + k_1$$

is Hurwitz

$$\alpha_1(\|z\|) \le V(z) \le \alpha_2(\|z\|)$$

$$\frac{\partial V}{\partial \eta}\bar{f}_0(z,s) \le -\alpha_3(\|z\|), \qquad \forall\ \|z\| \ge \alpha_4(|s|)$$

We have converted the relative degree ρ system into a relative degree one system that satisfies the earlier assumptions

$$u = \psi(\xi) + v$$

$$\left| \frac{\bar{a}(z,s) + \bar{b}(z,s)\psi(\xi) + \bar{\delta}(t,z,s,\psi(\xi)+v)}{\bar{b}(z,s)} \right| \le \varrho(\xi) + \kappa_0 |v|$$

Left hand side equals

$$\left| \frac{\sum_{i=1}^{\rho-1} k_i \xi_{i+1} + a(\eta,\xi) + b(\eta,\xi)\psi(\xi) + \delta(t,\eta,\xi,\psi(\xi)+v)}{b(\eta,\xi)} \right|$$

$$\beta(\xi) \ge \frac{\varrho(\xi)}{1-\kappa_0} + \beta_0, \quad \beta_0 > 0$$

$$u = \psi(\xi) - \beta(\xi)\,\mathrm{sat}\left(\frac{s}{\mu}\right) = \gamma(\xi)$$

Saturation Scheme 1:

$$M_i = \max_{\Omega}\{|\xi_i|\}, \quad 1 \le i \le \rho$$

$$\psi_s(\xi) = \left. \psi(\xi) \right|_{\xi_i = M_i \ \mathrm{sat}\left(\frac{\xi_i}{M_i}\right)} \qquad \beta_s(\xi) = \left. \beta(\xi) \right|_{\xi_i = M_i \ \mathrm{sat}\left(\frac{\xi_i}{M_i}\right)}$$

Scheme 2:

$$M_\psi = \max_{\Omega}\{|\psi(\xi)|\}, \qquad M_\beta = \max_{\Omega}\{|\beta(\xi)|\}$$

$$\psi_s(\xi) = M_\psi \ \mathrm{sat}(\psi(\xi)/M_\psi), \quad \beta_s(\xi) = M_\beta \ \mathrm{sat}(\beta(\xi)/M_\beta)$$

$$u = \psi_s(\xi) - \beta_s(\xi) \ \mathrm{sat}\left(\frac{s}{\mu}\right)$$

β_s and ψ_s are globally bounded functions of ξ

Scheme 3:

$$M_u = \max_{\Omega}\{|\psi(\xi) - \beta(x) \ \mathrm{sat}(s/\mu)|\}$$

$$u = M_u \ \mathrm{sat}\left(\frac{\psi(\xi) - \beta(\xi) \ \mathrm{sat}(s/\mu)}{M_u}\right)$$

Output Feedback Controller

$$\begin{aligned} \dot{\hat{\xi}}_i &= \hat{\xi}_{i+1} + \frac{\alpha_i}{\varepsilon^i}(y - \hat{\xi}_1), \quad 1 \le i \le \rho - 1 \\ \dot{\hat{\xi}}_\rho &= a_0(\hat{\xi}) + b_0(\hat{\xi})u + \frac{\alpha_\rho}{\varepsilon^\rho}(y - \hat{\xi}_1) \end{aligned}$$

$$s^\rho + \alpha_1 s^{\rho-1} + \cdots + \alpha_{\rho-1} s + \alpha_\rho \quad \text{is Hurwitz}$$

$$u = \gamma_s(\hat{\xi}), \ \gamma_s(\hat{\xi}) \text{ is given by}$$

$$\psi_s(\hat{\xi}) - \beta_s(\hat{\xi}) \ \mathrm{sat}(\frac{\hat{s}}{\mu}) \quad \text{or} \quad M_u \ \mathrm{sat}\left(\frac{\psi(\hat{\xi}) - \beta(\hat{\xi}) \ \mathrm{sat}(\hat{s}/\mu)}{M_u}\right)$$

$$\hat{s} = \sum_{i=1}^{\rho-1} k_i \hat{\xi}_i + \hat{\xi}_\rho$$

Theorem 12.5

Let Ω_0 be a compact set in the interior of Ω, X a compact subset of R^ρ, $(\eta(0), \xi(0)) \in \Omega_0$, and $\hat{\xi}(0) \in X$. Then, there exists ε^*, dependent on μ, such that for all $\varepsilon \in (0, \varepsilon^*)$ the states $(\eta(t), \xi(t), \hat{\xi}(t))$ of the closed-loop system are bounded for all $t \geq 0$ and there is a finite time T, dependent on μ, such that $(\eta(t), \xi(t)) \in \Omega_\mu$ for all $t \geq T$. Moreover, given any $\lambda > 0$, there exists $\varepsilon^{**} > 0$, dependent on μ and λ, such that for all $\varepsilon \in (0, \varepsilon^{**})$,

$$\|\eta(t) - \eta_r(t)\| \leq \lambda \quad \text{and} \quad \|\xi(t) - \xi_r(t)\| \leq \lambda, \quad \forall\ t \in [0, T]$$

where $(\eta_r(t), \xi_r(t))$ is the state of the closed-loop system under state feedback with initial conditions $\eta_r(0) = \eta(0)$ and $\xi_r(0) = \xi(0)$

Theorem 12.6

Suppose all the assumptions of Theorem 12.5 are satisfied. Then, there exists $\mu^* > 0$ and for each $\mu \in (0, \mu^*)$ there exists $\varepsilon^* > 0$, dependent on μ, such that for all $\varepsilon \in (0, \varepsilon^*)$, the origin of the closed-loop system under output feedback is exponentially stable and $\Omega_0 \times X$ is a subset of its region of attraction

Example 12.4 (Pendulum Equation)

$$\ddot{\theta} + \sin\theta + b\dot{\theta} = cu$$

$$0 \le b \le 0.2, \quad 0.5 \le c \le 2$$

From Example 10.1: $u = -2(|x_1| + |x_2| + 1)\ \mathrm{sat}\,(s/\mu)$

stabilizes the pendulum at $(\theta = \pi, \dot{\theta} = 0)$
Suppose now that we only measure θ. In preparation for using a high-gain observer, we will saturate the state feedback control outside a compact set

$$\Omega = \{|x_1| \le c/\theta_1\} \times \{|s| \le c\}, \quad c > 0,\ 0 < \theta_1 < 1$$

Take $c = 2\pi$ and $\theta = 0.8$

$$|x_1| \le 2.5\pi, \quad |x_2| \le 4.5\pi, \quad \forall\ x \in \Omega$$

Output Feedback Controller:

$$u = -2\left(2.5\pi\ \mathrm{sat}\left(\frac{|\hat{x}_1|}{2.5\pi}\right) + 4.5\pi\ \mathrm{sat}\left(\frac{|\hat{x}_2|}{4.5\pi}\right) + 1\right)\ \mathrm{sat}\left(\frac{\hat{s}}{\mu}\right)$$

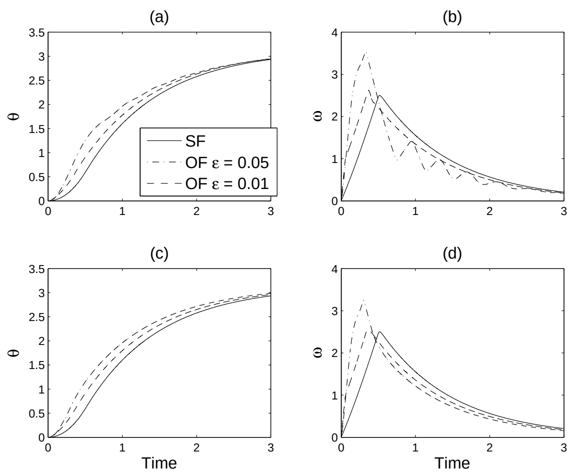
$$\hat{s} = \hat{x}_1 + \hat{x}_2$$

$$\dot{\hat{x}}_1 = \hat{x}_2 + (2/\varepsilon)(x_1 - \hat{x}_1), \quad \dot{\hat{x}}_2 = \phi_0(\hat{x}, u) + (1/\varepsilon^2)(x_1 - \hat{x}_1)$$

Simulation:

$$b = 0.01,\ c = 0.5, x_1(0) = -\pi,\ x_2(0) = \hat{x}_i(0) = 0$$

$$\phi_0(\hat{x}) = \begin{cases} 0 & \text{Figures (a) \& (b)} \\ -\sin(\hat{x}_1 + \pi) - 0.1\hat{x}_2 + 1.25u & \text{Figures (c) \& (d)} \end{cases}$$


(a)
(b)
(c)
(d)
θ
ω
Time
SF
OF ε = 0.05
OF ε = 0.01
0
0.5
1
1.5
2
2.5
3
3.5
4