

Nonlinear Control

Lecture # 36

Tracking & Regulation

Normal form:

$$\begin{aligned}\dot{\eta} &= f_0(\eta, \xi) \\ \dot{\xi}_i &= \xi_{i+1}, \quad \text{for } 1 \leq i \leq \rho - 1 \\ \dot{\xi}_\rho &= a(\eta, \xi) + b(\eta, \xi)u \\ y &= \xi_1\end{aligned}$$

$$\eta \in D_\eta \subset R^{n-\rho}, \xi = \text{col}(\xi_1, \dots, \xi_\rho) \in D_\xi \subset R^\rho$$

Tracking Problem: Design a feedback controller such that

$$\lim_{t \rightarrow \infty} [y(t) - r(t)] = 0$$

while ensuring boundedness of all state variables

Regulation Problem: r is constant

Assumption 13.1

$$b(\eta, \xi) \geq b_0 > 0, \quad \forall \eta \in D_\eta, \xi \in D_\xi$$

Assumption 13.2

$\dot{\eta} = f_0(\eta, \xi)$ is bounded-input–bounded-state stable over $D_\eta \times D_\xi$

Assumption 13.2 holds locally if the system is minimum phase and globally if $\dot{\eta} = f_0(\eta, \xi)$ is ISS

Assumption 13.3

$r(t)$ and its derivatives up to $r^{(\rho)}(t)$ are bounded for all $t \geq 0$ and the ρ th derivative $r^{(\rho)}(t)$ is a piecewise continuous function of t . Moreover, $\mathcal{R} = \text{col}(r, \dot{r}, \dots, r^{(\rho-1)}) \in D_\xi$ for all $t \geq 0$

The reference signal $r(t)$ could be specified as given functions of time, or it could be the output of a reference model

Example: For $\rho = 2$

$$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \quad \zeta > 0, \omega_n > 0$$

$$\dot{y}_1 = y_2, \quad \dot{y}_2 = -\omega_n^2 y_1 - 2\zeta\omega_n y_2 + \omega_n^2 u_c, \quad r = y_1$$

$$\dot{r} = y_2, \quad \ddot{r} = \dot{y}_2$$

Assumption 13.3 is satisfied when $u_c(t)$ is piecewise continuous and bounded

Change of variables:

$$e_1 = \xi_1 - r, \quad e_2 = \xi_2 - r^{(1)}, \quad \dots, \quad e_\rho = \xi_\rho - r^{(\rho-1)}$$

$$\begin{aligned}\dot{\eta} &= f_0(\eta, \xi) \\ \dot{e}_i &= e_{i+1}, \quad \text{for } 1 \leq i \leq \rho - 1 \\ \dot{e}_\rho &= a(\eta, \xi) + b(\eta, \xi)u - r^{(\rho)}\end{aligned}$$

Goal: Ensure $e = \text{col}(e_1, \dots, e_\rho) = \xi - \mathcal{R}$ is bounded for all $t \geq 0$ and converges to zero as t tends to infinity

Assumption 13.4

$r, r^{(1)}, \dots, r^{(\rho)}$ are available to the controller (needed in state feedback control)

Feedback controllers for tracking and regulation are classified as in stabilization

- State versus output feedback
- Static versus dynamic controllers
- Region of validity
 - local tracking
 - regional tracking
 - semiglobal tracking
 - global tracking

Local tracking is achieved for sufficiently small initial states and sufficiently small $\|\mathcal{R}\|$, while global tracking is achieved for any initial state and any bounded $\mathcal{R}$.

Practical tracking: The tracking error is ultimately bounded and the ultimate bound can be made arbitrarily small by choice of design parameters

- local practical tracking
- regional practical tracking
- semiglobal practical tracking
- global practical tracking

Tracking

$$\dot{\eta} = f_0(\eta, \xi), \quad \dot{e} = A_c e + B_c [a(\eta, \xi) + b(\eta, \xi)u - r^{(\rho)}]$$

Feedback linearization:

$$u = [-a(\eta, \xi) + r^{(\rho)} + v] / b(\eta, \xi)$$

$$\dot{\eta} = f_0(\eta, \xi), \quad \dot{e} = A_c e + B_c v$$

$$v = -K e, \quad A_c - B_c K \text{ is Hurwitz}$$

$$\dot{\eta} = f_0(\eta, \xi), \quad \dot{e} = (A_c - B_c K)e$$

$$A_c - B_c K \text{ Hurwitz} \Rightarrow e(t) \text{ is bounded and } \lim_{t \rightarrow \infty} e(t) = 0$$

$$\Rightarrow \xi = e + \mathcal{R} \text{ is bounded} \Rightarrow \eta \text{ is bounded}$$

Example 13.1 (Pendulum equation)

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\sin x_1 - bx_2 + cu, \quad y = x_1$$

We want the output y to track a reference signal $r(t)$

$$e_1 = x_1 - r, \quad e_2 = x_2 - \dot{r}$$

$$\dot{e}_1 = e_2, \quad \dot{e}_2 = -\sin x_1 - bx_2 + cu - \ddot{r}$$

$$u = \frac{1}{c}[\sin x_1 + bx_2 + \ddot{r} - k_1 e_1 - k_2 e_2]$$

$K = [k_1, k_2]$ assigns the eigenvalues of $A_c - B_c K$ at desired locations in the open left-half complex plane

Simulation

$$r = \sin(t/3), \quad x(0) = \text{col}(\pi/2, 0)$$

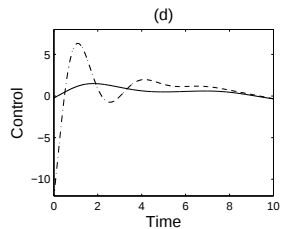
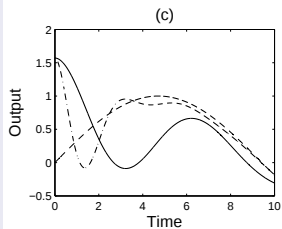
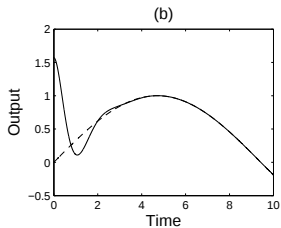
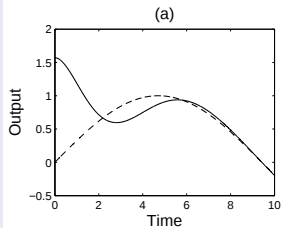
Nominal: $b = 0.03, c = 1$ Figures (a) and (b)

Perturbed: $b = 0.015, c = 0.5$ Figure (c)

Reference (dashed)

Low gain: $K = \begin{bmatrix} 1 & 1 \end{bmatrix}$, $\lambda = -0.5 \pm j0.5\sqrt{3}$, (solid)

High gain: $K = \begin{bmatrix} 9 & 3 \end{bmatrix}$, $\lambda = -1.5 \pm j1.5\sqrt{3}$, (dash-dot)



Robust Tracking

$$\begin{aligned}\dot{\eta} &= f_0(\eta, \xi) \\ \dot{e}_i &= e_{i+1}, \quad 1 \leq i \leq \rho - 1 \\ \dot{e}_\rho &= a(\eta, \xi) + b(\eta, \xi)u + \delta(t, \eta, \xi, u) - r^{(\rho)}(t)\end{aligned}$$

Sliding mode control: Design the sliding surface

$$\dot{e}_i = e_{i+1}, \quad 1 \leq i \leq \rho - 1$$

View e_ρ as the control input and design it to stabilize the origin

$$e_\rho = -(k_1 e_1 + \cdots + k_{\rho-1} e_{\rho-1})$$

$$\lambda^{\rho-1} + k_{\rho-1} \lambda^{\rho-2} + \cdots + k_1 \text{ is Hurwitz}$$

$$s = (k_1 e_1 + \cdots + k_{\rho-1} e_{\rho-1}) + e_\rho = 0$$

$$\dot{s} = \sum_{i=1}^{\rho-1} k_i e_{i+1} + a(\eta, \xi) + b(\eta, \xi)u + \delta(t, \eta, \xi, u) - r^{(\rho)}(t)$$

$$u = v \text{ or } u = -\frac{1}{\hat{b}(\eta, \xi)} \left[\sum_{i=1}^{\rho-1} k_i e_{i+1} + \hat{a}(\eta, \xi) - r^{(\rho)}(t) \right] + v$$

$$\dot{s} = b(\eta, \xi)v + \Delta(t, \eta, \xi, v)$$

$$\text{Suppose } \left| \frac{\Delta(t, \eta, \xi, v)}{b(\eta, \xi)} \right| \leq \varrho(\eta, \xi) + \kappa_0 |v|, \quad 0 \leq \kappa_0 < 1$$

$$v = -\beta(\eta, \xi) \operatorname{sat} \left(\frac{s}{\mu} \right), \quad \beta(\eta, \xi) \geq \frac{\varrho(\eta, \xi)}{(1 - \kappa_0)} + \beta_0, \quad \beta_0 > 0$$

$$s\dot{s} \leq -\beta_0 b_0(1 - \kappa_0)|s|, \quad |s| \geq \mu$$

$$\zeta = \text{col}(e_1, \dots, e_{\rho-1}), \quad \dot{\zeta} = \underbrace{(A_c - B_c K)}_{\text{Hurwitz}} \zeta + B_c s$$

$$V_0 = \zeta^T P \zeta, \quad P(A_c - B_c K) + (A_c - B_c K)^T P = -I$$

$$\dot{V}_0 = -\zeta^T \zeta + 2\zeta^T P B_c s \leq -(1-\theta)\|\zeta\|^2, \quad \forall \|\zeta\| \geq 2\|PB_c\| |s|/\theta$$

$$0 < \theta < 1. \quad \text{For } \sigma \geq \mu$$

$$\{\|\zeta\| \leq 2\|PB_c\| \sigma/\theta\} \subset \{\zeta^T P \zeta \leq \lambda_{\max}(P)(2\|PB_c\|/\theta)^2 \sigma^2\}$$

$$\rho_1 = \lambda_{\max}(P)(2\|PB_c\|/\theta)^2, \quad c > \mu$$

$$\Omega = \{\zeta^T P \zeta \leq \rho_1 c^2\} \times \{|s| \leq c\} \text{ is positively invariant}$$

For all $e(0) \in \Omega$, $e(t)$ enters the positively invariant set

$$\Omega_\mu = \{\zeta^T P \zeta \leq \rho_1 \mu^2\} \times \{|s| \leq \mu\}$$

Inside Ω_μ ,

$$|e_1| \leq k\mu$$

$$k = \|LP^{-1/2}\| \sqrt{\rho_1}, \quad L = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}$$

Example 13.2 (Reconsider Example 13.1)

$$\dot{e}_1 = e_2, \quad \dot{e}_2 = -\sin x_1 - bx_2 + cu - \ddot{r}$$

$$r(t) = \sin(t/3), \quad 0 \leq b \leq 0.1, \quad 0.5 \leq c \leq 2$$

$$s = e_1 + e_2$$

$$\begin{aligned} \dot{s} &= e_2 - \sin x_1 - bx_2 + cu - \ddot{r} \\ &= (1-b)e_2 - \sin x_1 - b\dot{r} - \ddot{r} + cu \end{aligned}$$

$$\left| \frac{(1-b)e_2 - \sin x_1 - b\dot{r} - \ddot{r}}{c} \right| \leq \frac{|e_2| + 1 + 0.1/3 + 1/9}{0.5}$$

$$u = -(2|e_2| + 3) \operatorname{sat} \left(\frac{e_1 + e_2}{\mu} \right)$$

Simulation:

$$\mu = 0.1, \quad x(0) = \text{col}(\pi/2, 0)$$

$$b = 0.03, \quad c = 1 \text{ (solid)}$$

$$b = 0.015, \quad c = 0.5 \text{ (dash-dot)}$$

Reference (dashed)

