Nonlinear Control

Lecture # 37

Tracking & Regulation

Transition Between Set Points

$$
\begin{aligned}
\dot{\eta} &= f_0(\eta, \xi) \\
\dot{\xi}_i &= \xi_{i+1}, \quad 1 \le i \le \rho - 1 \\
\dot{\xi}_\rho &= a(\eta, \xi) + b(\eta, \xi)u \\
y &= \xi_1
\end{aligned}
$$

Equilibrium point:

$$
\begin{aligned}
0 &= f_0(\bar{\eta}, \bar{\xi}) \\
0 &= \bar{\xi}_{i+1}, \quad 1 \le i \le \rho - 1 \\
0 &= a(\bar{\eta}, \bar{\xi}) + b(\bar{\eta}, \bar{\xi})\bar{u} \\
\bar{y} &= \bar{\xi}_1
\end{aligned}
$$

$$\bar{\xi}_1 = \bar{y}, \qquad \bar{\xi}_i = 0 \quad \text{for } 2 \le i \le \rho - 1$$

$$\bar{\xi} = \text{col}(\bar{y}, 0, \cdots, 0)$$

$$0 = f_0(\bar{\eta}, \bar{\xi}), \qquad \bar{u} = -\frac{a(\bar{\eta}, \bar{\xi})}{b(\bar{\eta}, \bar{\xi})}$$

Assume $0 = f_0(\bar{\eta}, \bar{\xi})$ has a unique solution $\bar{\eta}$ in the domain of interest

$$\bar{\eta} = \phi_\eta(\bar{y}), \qquad \bar{u} = \phi_u(\bar{y})$$

By assumption (and without loss of generality)

$$\phi_\eta(0) = 0, \qquad \phi_u(0) = 0$$

Goal: Move the system from equilibrium at $y = 0$ to equilibrium at $y = y^*$, either asymptotically or over a finite time period

First Approach: Apply a step command

$$r(t) = u_c(t) = \begin{cases} 0, & \text{for } t < 0 \\ y^* & \text{for } t \geq 0 \end{cases}$$

Is this allowed ?

$$\text{Take} \quad r = y^*, \quad \text{for } t \geq 0$$

$$r^{(i)} = 0 \;\; \text{for } i \geq 2$$

$$\eta(0) = 0, \quad e_1(0) = -y^*, \quad e_i(0) = 0 \;\; \text{for } i \geq 2$$

The shape of the transient response depends on the solution of

$$\dot{e} = (A_c - B_c K)e$$

in feedback linearization

or the solution of

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \vdots \\ \dot{e}_{\rho-1} \end{bmatrix} = \begin{bmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ -k_1 & & & -k_{\rho-1} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_{\rho-1} \end{bmatrix} + \begin{bmatrix} \\ \\ 1 \end{bmatrix} s$$

in sliding mode control

What is the impact of the reaching phase?

Second Approach: Take $r(t)$ as the zero-state response of a Hurwitz transfer function driven by u_c

Typical Choice:

$$\frac{a_\rho}{s^\rho + a_1 s^{\rho-1} + \cdots + a_{\rho-1}s + a_\rho}$$

Choose the parameters a_1 to a_ρ to shape the response of r

$$r(0) = 0 \quad \Rightarrow \quad e_1(0) = 0 \quad \Rightarrow \quad e(0) = 0$$

Feedback Linearization:

$$e(0) = 0 \quad \Rightarrow \quad e(t) \equiv 0$$

Sliding Mode Control:

$$e(0) = 0 \quad \Rightarrow \quad e(0) \in \Omega_\mu \quad \Rightarrow \quad e(t) \in \Omega_\mu, \; \forall \; t \geq 0$$

The derivatives of r are generated by the pre-filter

$$
\begin{aligned}
\dot{z} &= \begin{bmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ -a_\rho & & & -a_1 \end{bmatrix} z + \begin{bmatrix} \\ \\ \\ a_\rho \end{bmatrix} u_c \\
r &= \begin{bmatrix} 1 & & & \end{bmatrix} z
\end{aligned}
$$

$$r = z_1, \quad \dot{r} = z_2, \ldots\ldots \quad r^{(\rho-1)} = z_\rho$$

$$r^{(\rho)} = -\sum_{i=1}^{\rho} a_{\rho-i+1} z_i + a_\rho u_c$$

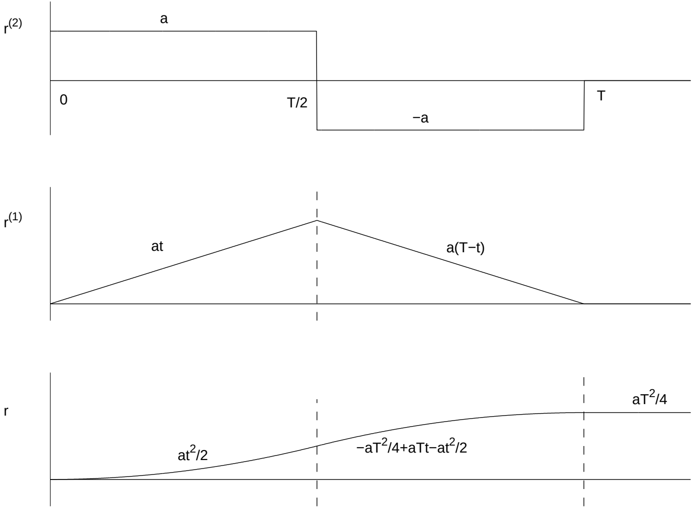
Does $r(t)$ satisfy the assumptions imposed last lecture?

Third Approach: Plan a trajectory $(r(t), \dot{r}(t), \ldots, r^{(\rho)}(t))$ to move from $(0, 0, \ldots, 0)$ to $(y^*, 0, \ldots, 0)$ in finite time T

Example: $\rho = 2$

$$r(t) = \begin{cases} \frac{at^2}{2} & \text{for } 0 \le t \le \frac{T}{2} \\ -\frac{aT^2}{4} + aTt - \frac{at^2}{2} & \text{for } \frac{T}{2} \le t \le T \\ \frac{aT^2}{4} & \text{for } t \ge T \end{cases}$$

$$a = \frac{4y^*}{T^2} \quad \Rightarrow \quad r(t) = y^* \quad \text{for } t \ge T$$


r(2)
a
0
T/2
−a
T
r(1)
at
a(T−t)
r
at2/2
−aT2/4+aTt−at2/2
aT2/4

Example 12.3 (Reconsider Example 13.1)

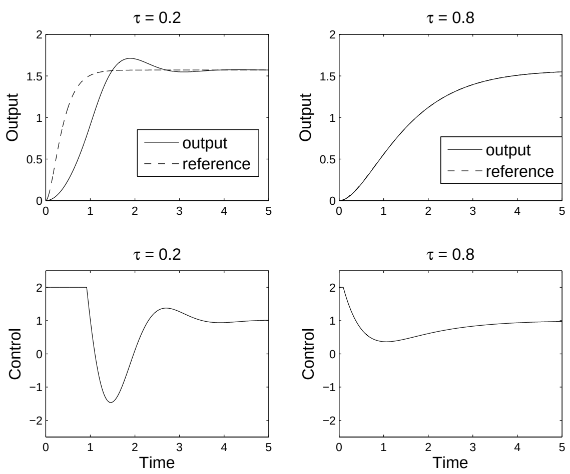
$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\sin x_1 - 0.03x_2 + u, \quad y = x_1$$

Move the pendulum from equilibrium at $x = 0$ to equilibrium at $x = \text{col}(\pi/2, 0)$ Pre-Filter:

$$\frac{1}{(\tau s + 1)^2}, \quad u_c = \begin{cases} 0, & \text{for } t < 0 \\ \frac{\pi}{2} & \text{for } t \geq 0 \end{cases}$$

$$u = \sin x_1 + 0.03x_2 + \ddot{r} - 9e_1 - 3e_2$$

$$\color{red}{Constraint:} \quad |u(t)| \leq 2$$


τ = 0.2
τ = 0.8
Output
output
reference
τ = 0.2
τ = 0.8
Control
Time

Robust Regulation via Integral Action

$$
\begin{aligned}
\dot{\eta} &= f_0(\eta, \xi, w) \\
\dot{\xi}_i &= \xi_{i+1}, \qquad 1 \le i \le \rho - 1 \\
\dot{\xi}_\rho &= a(\eta, \xi, w) + b(\eta, \xi, w)u \\
y &= \xi_1
\end{aligned}
$$

Disturbance w and reference r are constant

Equilibrium point:

$$
\begin{aligned}
0 &= f_0(\bar{\eta}, \bar{\xi}, w) \\
0 &= \bar{\xi}_{i+1}, \qquad 1 \le i \le \rho - 1 \\
0 &= a(\bar{\eta}, \bar{\xi}, w) + b(\bar{\eta}, \bar{\xi}, w)\bar{u} \\
r &= \bar{\xi}_1
\end{aligned}
$$

Assumption 13.5

$0 = f_0(\bar{\eta}, \bar{\xi}, w)$ has a unique solution $\bar{\eta} = \phi_\eta(r, w)$

$$\bar{u} = -\,\frac{a(\bar{\eta}, \bar{\xi}, w)}{b(\bar{\eta}, \bar{\xi}, w)} \stackrel{\text{def}}{=} \phi_u(r, w)$$

Augment the integrator $\quad \dot{e}_0 = y - r$

$$z = \eta - \bar{\eta}, \qquad e = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_\rho \end{bmatrix} = \begin{bmatrix} \xi_1 - r \\ \xi_2 \\ \vdots \\ \xi_\rho \end{bmatrix}$$

$$
\begin{aligned}
\dot{z} &= f_0(z+\bar{\eta},\xi,w) \stackrel{\text{def}}{=} \tilde{f}_0(z,e,r,w)\\
\dot{e}_i &= e_{i+1}, \qquad \text{for } 0 \le i \le \rho-1\\
\dot{e}_\rho &= a(\eta,\xi,w)+b(\eta,\xi,w)u
\end{aligned}
$$

Sliding mode control:

$$s = k_0e_0 + k_1e_1 + \cdots + k_{\rho-1}e_{\rho-1} + e_\rho$$

$$\lambda^\rho + k_{\rho-1}\lambda^{\rho-1} + \cdots + k_1\lambda + k_0 \quad \text{is Hurwitz}$$

$$\dot{s} = \sum_{i=0}^{\rho-1} k_ie_{i+1} + a(\eta,\xi,w) + b(\eta,\xi,w)u$$

$$u = v \quad \text{or} \quad u = -\frac{1}{\hat{b}(\eta,\xi)}\left[\sum_{i=0}^{\rho-1} k_ie_{i+1} + \hat{a}(\eta,\xi)\right] + v$$

$$\dot{s} = b(\eta, \xi, w)v + \Delta(\eta, \xi, r, w)$$

$$\left| \frac{\Delta(\eta, \xi, r, w)}{b(\eta, \xi, w)} \right| \leq \varrho(\eta, \xi)$$

$$v = -\beta(\eta, \xi)\ \mathrm{sat}\left(\frac{s}{\mu}\right), \quad \beta(\eta, \xi) \geq \varrho(\eta, \xi) + \beta_0,\ \ \beta_0 > 0$$

Assumption 13.6

$$\alpha_1(\|z\|) \leq V_1(z, r, w) \leq \alpha_2(\|z\|)$$

$$\frac{\partial V_1}{\partial z} \tilde{f}_0(z, e, r, w) \leq -\alpha_3(\|z\|), \qquad \forall\ \|z\| \geq \alpha_4(\|e\|)$$

Assumption 13.7

$z = 0$ is an exponentially stable equilibrium point of
$\dot{z} = \tilde{f}_0(z, 0, r, w)$

Theorem 13.1

Under the stated assumptions, there are positive constants c, ρ_1 and ρ_2 and a positive definite matrix P such that the set

$$\Omega = \{V_1(z) \le \alpha_2(\alpha_4(c\rho_2))\} \times \{\zeta^T P \zeta \le \rho_1 c^2\} \times \{|s| \le c\}$$

where $\zeta = \mathrm{col}(e_0, e_1, \ldots, e_{\rho-1})$, is compact and positively invariant, and for all initial states in Ω

$$\lim_{t\to\infty} |y(t) - r| = 0$$

Special case: $\beta = k$ (a constant) and $u = v$

$$u = -k \ \mathrm{sat}\left(\frac{k_0 e_0 + k_1 e_1 + \cdots + k_{\rho-1} e_{\rho-1} + e_\rho}{\mu}\right)$$

Example 13.4 (Pendulum with horizontal acceleration)

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\sin x_1 - bx_2 + cu + d\cos x_1, \quad y = x_1$$

d is constant. Regulate y to a constant reference r

$$0 \le b \le 0.1, \quad 0.5 \le c \le 2, \quad 0 \le d \le 0.5$$

$$e_1 = x_1 - r, \quad e_2 = x_2$$

$$\dot{e}_0 = e_1, \quad \dot{e}_1 = e_2, \quad \dot{e}_2 = -\sin x_1 - bx_2 + cu + d\cos x_1$$

$$s = e_0 + 2e_1 + e_2$$

$$\dot{s} = e_1 + (2-b)e_2 - \sin x_1 + cu + d\cos x_1$$

$$\left|\frac{e_1 + (2-b)e_2 - \sin x_1 + d\cos x_1}{c}\right| \le \frac{|e_1| + 2|e_2| + 1 + 0.5}{0.5}$$

$$u = -(2|e_1| + 4|e_2| + 4)\ \mathrm{sat}\left(\frac{e_0 + 2e_1 + e_2}{\mu}\right)$$

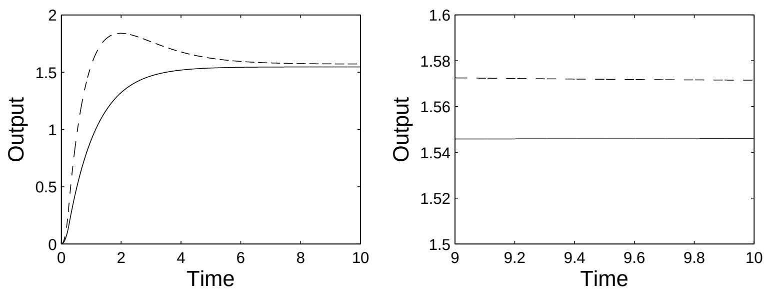
For comparison, SMC without integrator

$$s = e_1 + e_2, \quad \dot{s} = (1-b)e_2 - \sin x_1 + cu + d\cos x_1$$

$$u = -(2|e_2| + 4)\ \mathrm{sat}\left(\frac{e_1 + e_2}{\mu}\right)$$

Simulation: With integrator (dashed), without (solid)

$$\mu = 0.1,\ x(0) = 0,\ r = \pi/2,\ b = 0.03,\ c = 1,\ d = 0.3$$



Output
Time
0
0.5
1
1.5
2
0
2
4
6
8
10
1.5
1.52
1.54
1.56
1.58
1.6
9
9.2
9.4
9.6
9.8
10