Nonlinear Control

Lecture # 6

Stability of Equilibrium Points

The Invariance Principle

Example: Pendulum equation with friction

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\,a\sin x_1 - bx_2$$

$$V(x) = a(1-\cos x_1) + \frac{1}{2}x_2^2$$

$$\dot{V}(x) = a\dot{x}_1 \sin x_1 + x_2\dot{x}_2 = -\,bx_2^2$$

The origin is stable. $\dot{V}(x)$ is not negative definite because $\dot{V}(x) = 0$ for $x_2 = 0$ irrespective of the value of x_1

However, near the origin, the solution cannot stay identically in the set $\{x_2 = 0\}$

Definitions

Let $x(t)$ be a solution of $\dot{x} = f(x)$

A point p is a *positive limit point* of $x(t)$ if there is a sequence $\{t_n\}$, with $\lim_{n\to\infty} t_n = \infty$, such that $x(t_n) \to p$ as $n \to \infty$

The set of all positive limit points of $x(t)$ is called the *positive limit set* of $x(t)$; denoted by L^+

If $x(t)$ approaches an asymptotically stable equilibrium point $\bar{x}$, then $\bar{x}$ is the positive limit point of $x(t)$ and $L^+ = \bar{x}$

A stable limit cycle is the positive limit set of every solution starting sufficiently near the limit cycle

A set M is an *invariant set* with respect to $\dot{x} = f(x)$ if

$$x(0) \in M \Rightarrow x(t) \in M, \quad \forall\, t \in R$$

Examples:

- Equilibrium points
- Limit Cycles

A set M is a *positively invariant set* with respect to $\dot{x} = f(x)$ if

$$x(0) \in M \Rightarrow x(t) \in M, \quad \forall\, t \geq 0$$

Example; The set $\Omega_c = \{V(x) \leq c\}$ with $\dot{V}(x) \leq 0$ in Ω_c

The distance from a point p to a set M is defined by

$$\operatorname{dist}(p, M) = \inf_{x \in M} \|p - x\|$$

$x(t)$ approaches a set M as t approaches infinity, if for each $\varepsilon > 0$ there is $T > 0$ such that

$$\operatorname{dist}(x(t), M) < \varepsilon, \ \ \forall \ t > T$$

Example: every solution $x(t)$ starting sufficiently near a stable limit cycle approaches the limit cycle as $t \to \infty$

Notice, however, that $x(t)$ does converge to any specific point on the limit cycle

Lemma 3.1

If a solution $x(t)$ of $\dot{x} = f(x)$ is bounded and belongs to D for $t \geq 0$, then its positive limit set L^+ is a nonempty, compact, invariant set. Moreover, $x(t)$ approaches L^+ as $t \to \infty$

LaSalle's Theorem (3.4)

Let $f(x)$ be a locally Lipschitz function defined over a domain $D \subset R^n$ and $\Omega \subset D$ be a compact set that is positively invariant with respect to $\dot{x} = f(x)$. Let $V(x)$ be a continuously differentiable function defined over D such that $\dot{V}(x) \leq 0$ in Ω. Let E be the set of all points in Ω where $\dot{V}(x) = 0$, and M be the largest invariant set in E. Then every solution starting in Ω approaches M as $t \to \infty$

Proof

$$\dot{V}(x) \le 0 \ \text{ in } \ \Omega \quad \Rightarrow \quad V(x(t)) \text{ is a decreasing}$$

$$V(x) \text{ is continuous in } \Omega \quad \Rightarrow \quad V(x) \ge b = \min_{x\in\Omega} V(x)$$

$$\Rightarrow \quad \lim_{t\to\infty} V(x(t)) = a$$

$$x(t) \in \Omega \quad \Rightarrow \quad x(t) \text{ is bounded} \quad \Rightarrow \quad L^+ \text{ exists}$$

Moreover, $L^+ \subset \Omega$ and $x(t)$ approaches L^+ as $t \to \infty$

For any $p \in L^+$, there is $\{t_n\}$ with $\lim_{n\to\infty} t_n = \infty$ such that $x(t_n) \to p$ as $n \to \infty$

$$V(x) \text{ is continuous} \quad \Rightarrow \quad V(p) = \lim_{n\to\infty} V(x(t_n)) = a$$

$$V(x) = a \text{ on } L^+ \text{ and } L^+ \text{ invariant} \quad \Rightarrow \quad \dot{V}(x) = 0, \ \forall \ x \in L^+$$

$$L^+ \subset M \subset E \subset \Omega$$

$$x(t) \text{ approaches } L^+ \ \Rightarrow \ x(t) \text{ approaches } M \ (\text{as } t \to \infty)$$

Theorem 3.5

Let $f(x)$ be a locally Lipschitz function defined over a domain $D \subset R^n$; $0 \in D$. Let $V(x)$ be a continuously differentiable positive definite function defined over D such that $\dot{V}(x) \leq 0$ in D. Let $S = \{x \in D \mid \dot{V}(x) = 0\}$

- If no solution can stay identically in S, other than the trivial solution $x(t) \equiv 0$, then the origin is asymptotically stable
- Moreover, if $\Gamma \subset D$ is compact and positively invariant, then it is a subset of the region of attraction
- Furthermore, if $D = R^n$ and $V(x)$ is radially unbounded, then the origin is globally asymptotically stable

Example 3.8

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -h_1(x_1) - h_2(x_2)$$

$$h_i(0) = 0, \quad y h_i(y) > 0, \quad \text{for } 0 < |y| < a$$

$$V(x) = \int_0^{x_1} h_1(y)\ dy \ + \ \tfrac{1}{2}x_2^2$$

$$D = \{-a < x_1 < a, \quad -a < x_2 < a\}$$

$$\dot{V}(x) = h_1(x_1)x_2 + x_2[-h_1(x_1) - h_2(x_2)] = -x_2 h_2(x_2) \le 0$$

$$\dot{V}(x) = 0 \ \Rightarrow \ x_2 h_2(x_2) = 0 \ \Rightarrow \ x_2 = 0$$

$$S = \{x \in D \mid x_2 = 0\}$$

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -h_1(x_1) - h_2(x_2)$$

$$x_2(t) \equiv 0 \Rightarrow \dot{x}_2(t) \equiv 0 \Rightarrow h_1(x_1(t)) \equiv 0 \Rightarrow x_1(t) \equiv 0$$

The only solution that can stay identically in S is $x(t) \equiv 0$

Thus, the origin is asymptotically stable

Suppose $a = \infty$ and $\int_0^y h_1(z)\ dz\ \rightarrow\ \infty\ \text{ as }\ |y| \rightarrow\ \infty$

Then, $D = R^2$ and $V(x) = \int_0^{x_1} h_1(y)\ dy\ + \frac{1}{2}x_2^2$ is radially unbounded. $S = \{x \in R^2 \mid x_2 = 0\}$ and the only solution that can stay identically in S is $x(t) \equiv 0$

The origin is globally asymptotically stable

Exponential Stability

The origin of $\dot{x} = f(x)$ is exponentially stable if and only if the linearization of $f(x)$ at the origin is Hurwitz

Theorem 3.6

Let $f(x)$ be a locally Lipschitz function defined over a domain $D \subset R^n$; $0 \in D$. Let $V(x)$ be a continuously differentiable function such that

$$k_1\|x\|^a \le V(x) \le k_2\|x\|^a, \qquad \dot{V}(x) \le -k_3\|x\|^a$$

for all $x \in D$, where k_1, k_2, k_3, and a are positive constants. Then, the origin is an exponentially stable equilibrium point of $\dot{x} = f(x)$. If the assumptions hold globally, the origin will be globally exponentially stable

Proof

Choose $c > 0$ small enough that $\{k_1\|x\|^a \leq c\} \subset D$

$$V(x) \leq c \;\Rightarrow\; k_1\|x\|^a \leq c$$

$$\Omega_c = \{V(x) \leq c\} \subset \{\; k_1\|x\|^a \leq c\} \subset D$$

Ω_c is compact and positively invariant; $\forall\; x(0) \in \Omega_c$

$$\dot{V} \leq -k_3\|x\|^a \;\; \leq -\;\frac{k_3}{k_2}V$$

$$\frac{dV}{V} \leq -\;\frac{k_3}{k_2}\;dt$$

$$V(x(t)) \leq V(x(0))e^{-(k_3/k_2)t}$$

$$
\begin{aligned}
\|x(t)\| &\le \left[\frac{V(x(t))}{k_1}\right]^{1/a} \\
&\le \left[\frac{V(x(0))e^{-(k_3/k_2)t}}{k_1}\right]^{1/a} \\
&\le \left[\frac{k_2\|x(0)\|^a e^{-(k_3/k_2)t}}{k_1}\right]^{1/a} \\
&= \left(\frac{k_2}{k_1}\right)^{1/a} e^{-\gamma t}\ \|x(0)\|, \qquad \gamma = k_3/(k_2 a)
\end{aligned}
$$

Example 3.10

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -h(x_1) - x_2$$

$$c_1 y^2 \leq y h(y) \leq c_2 y^2, \quad \forall\ y,\ c_1 > 0,\ c_2 > 0$$

$$V(x) = \tfrac{1}{2}\ x^T \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} x + 2\int_0^{x_1} h(y)\ dy$$

$$c_1 x_1^2 \leq 2\int_0^{x_1} h(y)\ dy \leq c_2 x_1^2$$

$$\begin{aligned} \dot{V} &= [x_1 + x_2 + 2h(x_1)]x_2 + [x_1 + 2x_2][-h(x_1) - x_2] \\ &= -x_1 h(x_1) - x_2^2 \ \leq\ -c_1 x_1^2 - x_2^2 \end{aligned}$$

The origin is globally exponentially stable

Quadratic Forms

$$V(x) = x^T P x = \sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij} x_i x_j, \quad P = P^T$$

$$\lambda_{\min}(P) \|x\|^2 \le x^T P x \le \lambda_{\max}(P) \|x\|^2$$

$$P \ge 0 \text{ (Positive semidefinite) if and only if } \lambda_i(P) \ge 0 \ \forall i$$

$$P > 0 \text{ (Positive definite) if and only if } \lambda_i(P) > 0 \ \forall i$$

$P > 0$ if and only if all the leading principal minors of P are positive

Linear Systems

$$\dot{x} = Ax$$

$$V(x) = x^T P x, \qquad\qquad P = P^T > 0$$

$$\dot{V}(x) = x^T P \dot{x} + \dot{x}^T P x = x^T (PA + A^T P) x \stackrel{\text{def}}{=} -x^T Q x$$

If $Q > 0$, then A is Hurwitz

Or choose $Q > 0$ and solve the Lyapunov equation

$$PA + A^T P = -Q$$

If $P > 0$, then A is Hurwitz

MATLAB: $P = \mathsf{lyap}(A', Q)$

Theorem 3.7

A matrix A is Hurwitz if and only if for every $Q = Q^T > 0$ there is $P = P^T > 0$ that satisfies the Lyapunov equation

$$PA + A^T P = -Q$$

Moreover, if A is Hurwitz, then P is the unique solution

Linearization

$$\dot{x} = f(x) = [A + G(x)]x$$

$$G(x) \to 0 \ \text{ as } \ x \to 0$$

Suppose A is Hurwitz. Choose $Q = Q^T > 0$ and solve $PA + A^T P = -Q$ for P. Use $V(x) = x^T Px$ as a Lyapunov function candidate for $\dot{x} = f(x)$

$$\begin{aligned}
\dot{V}(x) &= x^T Pf(x) + f^T(x)Px \\
&= x^T P[A + G(x)]x + x^T[A^T + G^T(x)]Px \\
&= x^T(PA + A^T P)x + 2x^T PG(x)x \\
&= -x^T Qx + 2x^T PG(x)x
\end{aligned}$$

$$\dot{V}(x) \le -x^T Q x + 2\|P\ G(x)\|\ \|x\|^2$$

Given any positive constant $k < 1$, we can find $r > 0$ such that

$$2\|PG(x)\| < k\lambda_{\min}(Q), \quad \forall\ \|x\| < r$$

$$x^T Q x \ge \lambda_{\min}(Q)\|x\|^2 \iff -x^T Q x \le -\lambda_{\min}(Q)\|x\|^2$$

$$\dot{V}(x) \le -(1-k)\lambda_{\min}(Q)\|x\|^2, \quad \forall\ \|x\| < r$$

$V(x) = x^T P x$ is a Lyapunov function for $\dot{x} = f(x)$