Nonlinear Control

Lecture # 7

Stability of Equilibrium Points

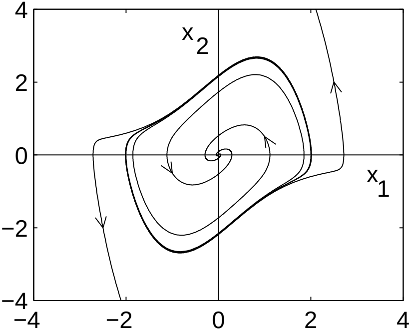
Region of Attraction

Lemma 3.2

The region of attraction of an asymptotically stable equilibrium point is an open, connected, invariant set, and its boundary is formed by trajectories

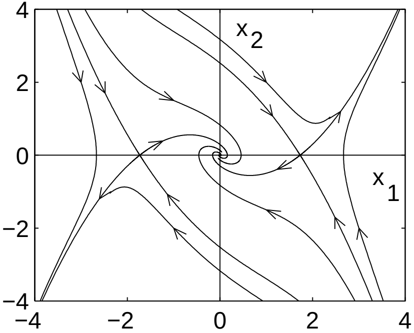
Example 3.11

$$\dot{x}_1 = -x_2, \qquad \dot{x}_2 = x_1 + (x_1^2 - 1)x_2$$



Example 3.12

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 + \tfrac{1}{3}x_1^3 - x_2$$



Estimates of the Region of Attraction: Find a subset of the region of attraction

Warning: Let D be a domain with $0 \in D$ such that for all $x \in D$, $V(x)$ is positive definite and $\dot{V}(x)$ is negative definite

Is D a subset of the region of attraction?

NO

Why?

Example 3.13

Reconsider

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 + \tfrac{1}{3}x_1^3 - x_2$$

$$\begin{aligned} V(x) &= \tfrac{1}{2}x^T \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} x + 2\int_0^{x_1} (y - \tfrac{1}{3}y^3)\ dy \\ &= \tfrac{3}{2}x_1^2 - \tfrac{1}{6}x_1^4 + x_1x_2 + x_2^2 \end{aligned}$$

$$\dot{V}(x) = -x_1^2(1 - \tfrac{1}{3}x_1^2) - x_2^2$$

$$D = \{-\sqrt{3} < x_1 < \sqrt{3}\}$$

Is D a subset of the region of attraction?

By Theorem 3.5, if D is a domain that contains the origin such that $\dot{V}(x) \leq 0$ in D, then the region of attraction can be estimated by a compact positively invariant set $\Gamma \in D$ if

- $\dot{V}(x) < 0$ for all $x \in \Gamma$, $x \neq 0$, or
- No solution can stay identically in $\{x \in D \mid \dot{V}(x) = 0\}$ other than the zero solution.

The simplest such estimate is the set $\Omega_c = \{V(x) \leq c\}$ when Ω_c is bounded and contained in D

$$V(x) = x^T P x, \quad P = P^T > 0, \quad \Omega_c = \{V(x) \le c\}$$

If $D = \{\|x\| < r\}$, then $\Omega_c \subset D$ if

$$c < \min_{\|x\|=r} x^T P x = \lambda_{\min}(P) r^2$$

If $D = \{|b^T x| < r\}$, where $b \in R^n$, then

$$\min_{|b^T x|=r} x^T P x = \frac{r^2}{b^T P^{-1} b}$$

Therefore, $\Omega_c \subset D = \{|b_i^T x| < r_i, \ i = 1, \ldots, p\}$, if

$$c < \min_{1 \le i \le p} \frac{r_i^2}{b_i^T P^{-1} b_i}$$

Example 3.14

$$\dot{x}_1 = -x_2, \qquad \dot{x}_2 = \; x_1 + (x_1^2 - 1)x_2$$

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

has eigenvalues $(-1 \pm j\sqrt{3})/2$. Hence the origin is asymptotically stable

Take $Q = I, \quad PA + A^T P = -I \;\Rightarrow\; P = \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 1 \end{bmatrix}$

$\lambda_{\min}(P) = 0.691$

$$V(x) = 1.5x_1^2 - x_1x_2 + x_2^2$$

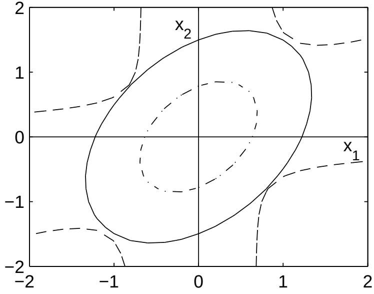
$$\dot{V}(x) = -(x_1^2 + x_2^2) - x_1^2x_2(x_1 - 2x_2)$$

$$|x_1| \le \|x\|, \quad |x_1x_2| \le \tfrac{1}{2}\|x\|^2, \quad |x_1 - 2x_2| \le \sqrt{5}\|x\|$$

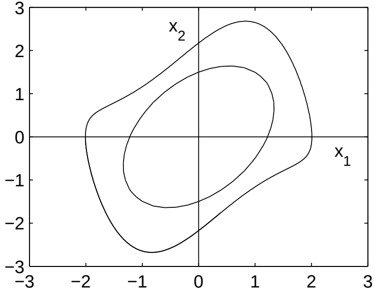
$$\dot{V}(x) \le -\|x\|^2 + \frac{\sqrt{5}}{2}\|x\|^4 < 0 \;\text{ for } 0 < \|x\|^2 < \frac{2}{\sqrt{5}} \stackrel{\text{def}}{=} r^2$$

$$\text{Take } \; c < \lambda_{\min}(P)r^2 = 0.691 \times \frac{2}{\sqrt{5}} = 0.618$$

$\{V(x) \le c\}$ is an estimate of the region of attraction



(a)



(b)

(a) Contours of $\dot{V}(x) = 0$ (dashed)
$V(x) = 0.618$ (dash-dot), $V(x) = 2.25$ (solid)
(b) comparison of the region of attraction with its estimate

Remark 3.1

If $\Omega_1, \Omega_2, \ldots, \Omega_m$ are positively invariant subsets of the region of attraction, then their union $\cup_{i=1}^{m} \Omega_i$ is also a positively invariant subset of the region of attraction. Therefore, if we have multiple Lyapunov functions for the same system and each function is used to estimate the region of attraction, we can enlarge the estimate by taking the union of all the estimates

Remark 3.2

we can work with any compact set $\Gamma \subset D$ provided we can show that Γ is positively invariant. This typically requires investigating the vector field at the boundary of Γ to ensure that trajectories starting in Γ cannot leave it

Example 3.15

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -4(x_1 + x_2) - h(x_1 + x_2)$$

$$h(0) = 0; \quad uh(u) \geq 0, \ \forall \ |u| \leq 1$$

$$V(x) = x^T P x = x^T \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} x = 2x_1^2 + 2x_1x_2 + x_2^2$$

$$\begin{aligned} \dot{V}(x) &= (4x_1 + 2x_2)\dot{x}_1 + 2(x_1 + x_2)\dot{x}_2 \\ &= -2x_1^2 - 6(x_1 + x_2)^2 - 2(x_1 + x_2)h(x_1 + x_2) \\ &\leq -2x_1^2 - 6(x_1 + x_2)^2, \quad \forall \ |x_1 + x_2| \leq 1 \\ &= -x^T \begin{bmatrix} 8 & 6 \\ 6 & 6 \end{bmatrix} x \end{aligned}$$

$$V(x) = x^T P x = x^T \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} x$$

$\dot{V}(x)$ is negative definite in $\{|x_1 + x_2| \leq 1\}$

$$b^T = [1 \quad 1], \qquad c = \min_{|x_1+x_2|=1} x^T P x = \frac{1}{b^T P^{-1} b} = 1$$

The region of attraction is estimated by $\{V(x) \leq 1\}$

$$\sigma = x_1 + x_2$$

$$\frac{d}{dt}\sigma^2 = 2\sigma x_2 - 8\sigma^2 - 2\sigma h(\sigma) \le 2\sigma x_2 - 8\sigma^2, \quad \forall\ |\sigma| \le 1$$

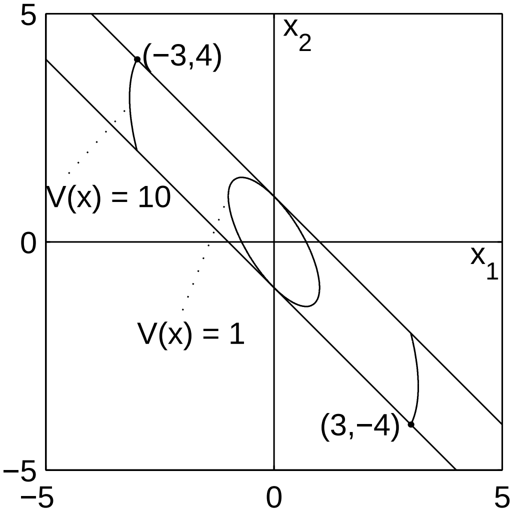
$$\text{On } \sigma = 1, \quad \frac{d}{dt}\sigma^2 \le 2x_2 - 8 \le 0, \quad \forall\ x_2 \le 4$$

$$\text{On } \sigma = -1, \quad \frac{d}{dt}\sigma^2 \le -2x_2 - 8 \le 0, \quad \forall\ x_2 \ge -4$$

$$c_1 = V(x)|_{x_1=-3, x_2=4} = 10, \qquad c_2 = V(x)|_{x_1=3, x_2=-4} = 10$$

$$\Gamma = \{V(x) \le 10 \text{ and } |x_1 + x_2| \le 1\}$$

is a subset of the region of attraction


5
0
−5
−5
0
5
x2
x1
(−3,4)
(3,−4)
V(x) = 10
V(x) = 1

Converse Lyapunov Theorems

Theorem 3.8 (Exponential Stability)

Let $x = 0$ be an exponentially stable equilibrium point for the system $\dot{x} = f(x)$, where f is continuously differentiable on $D = \{\|x\| < r\}$. Let k, λ, and r_0 be positive constants with $r_0 < r/k$ such that

$$\|x(t)\| \leq k\|x(0)\|e^{-\lambda t}, \quad \forall\ x(0) \in D_0,\ \forall\ t \geq 0$$

where $D_0 = \{\|x\| < r_0\}$. Then, there is a continuously differentiable function $V(x)$ that satisfies the inequalities

$$c_1\|x\|^2 \le V(x) \le c_2\|x\|^2$$

$$\frac{\partial V}{\partial x} f(x) \le -c_3\|x\|^2$$

$$\left\|\frac{\partial V}{\partial x}\right\| \le c_4\|x\|$$

for all $x \in D_0$, with positive constants c_1, c_2, c_3, and c_4
Moreover, if f is continuously differentiable for all x, globally Lipschitz, and the origin is globally exponentially stable, then $V(x)$ is defined and satisfies the aforementioned inequalities for all $x \in R^n$

Example 3.16

Consider the system $\dot{x} = f(x)$ where f is continuously differentiable in the neighborhood of the origin and $f(0) = 0$. Show that the origin is exponentially stable only if $A = [\partial f/\partial x](0)$ is Hurwitz

$$f(x) = Ax + G(x)x, \qquad G(x) \to 0 \text{ as } x \to 0$$

Given any $L > 0$, there is $r_1 > 0$ such that

$$\|G(x)\| \leq L, \;\; \forall \; \|x\| < r_1$$

Because the origin of $\dot{x} = f(x)$ is exponentially stable, let $V(x)$ be the function provided by the converse Lyapunov theorem over the domain $\{\|x\| < r_0\}$. Use $V(x)$ as a Lyapunov function candidate for $\dot{x} = Ax$

$$\begin{aligned}
\frac{\partial V}{\partial x}Ax &= \frac{\partial V}{\partial x}f(x) - \frac{\partial V}{\partial x}G(x)x \\
&\leq -c_3\|x\|^2 + c_4 L\|x\|^2 \\
&= -(c_3 - c_4 L)\|x\|^2
\end{aligned}$$

Take $L < c_3/c_4, \quad \gamma \stackrel{\text{def}}{=} (c_3 - c_4 L) > 0 \ \Rightarrow$

$$\frac{\partial V}{\partial x}Ax \leq -\gamma\|x\|^2, \quad \forall \ \|x\| < \min\{r_0, \ r_1\}$$

The origin of $\dot{x} = Ax$ is exponentially stable

Theorem 3.9 (Asymptotic Stability)

Let $x = 0$ be an asymptotically stable equilibrium point for $\dot{x} = f(x)$, where f is locally Lipschitz on a domain $D \subset R^n$ that contains the origin. Let $R_A \subset D$ be the region of attraction of $x = 0$. Then, there is a smooth, positive definite function $V(x)$ and a continuous, positive definite function $W(x)$, both defined for all $x \in R_A$, such that

$$V(x) \to \infty \text{ as } x \to \partial R_A$$

$$\frac{\partial V}{\partial x} f(x) \leq -W(x), \quad \forall\, x \in R_A$$

and for any $c > 0$, $\{V(x) \leq c\}$ is a compact subset of R_A
When $R_A = R^n$, $V(x)$ is radially unbounded