Nonlinear Control

Lecture # 8

Time Varying

and

Perturbed Systems

Time-varying Systems

$$\dot{x} = f(t, x)$$

$f(t, x)$ is piecewise continuous in t and locally Lipschitz in x for all $t \geq 0$ and all $x \in D$, $(0 \in D)$. The origin is an equilibrium point at $t = 0$ if

$$f(t, 0) = 0, \quad \forall\ t \geq 0$$

While the solution of the time-invariant system

$$\dot{x} = f(x), \qquad x(t_0) = x_0$$

depends only on $(t - t_0)$, the solution of

$$\dot{x} = f(t, x), \qquad x(t_0) = x_0$$

may depend on both t and t_0

Comparison Functions

- A scalar continuous function $\alpha(r)$, defined for $r \in [0, a)$, belongs to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It belongs to class $\mathcal{K}_\infty$ if it is defined for all $r \geq 0$ and $\alpha(r) \to \infty$ as $r \to \infty$
- A scalar continuous function $\beta(r, s)$, defined for $r \in [0, a)$ and $s \in [0, \infty)$, belongs to class $\mathcal{KL}$ if, for each fixed s, the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r and, for each fixed r, the mapping $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$

Example 4.1

- $\alpha(r) = \tan^{-1}(r)$ is strictly increasing since $\alpha'(r) = 1/(1+r^2) > 0$. It belongs to class $\mathcal{K}$, but not to class $\mathcal{K}_\infty$ since $\lim_{r\to\infty} \alpha(r) = \pi/2 < \infty$
- $\alpha(r) = r^c$, $c > 0$, is strictly increasing since $\alpha'(r) = cr^{c-1} > 0$. Moreover, $\lim_{r\to\infty} \alpha(r) = \infty$; thus, it belongs to class $\mathcal{K}_\infty$
- $\alpha(r) = \min\{r, r^2\}$ is continuous, strictly increasing, and $\lim_{r\to\infty} \alpha(r) = \infty$. Hence, it belongs to class $\mathcal{K}_\infty$. It is not continuously differentiable at $r = 1$. Continuous differentiability is not required for a class $\mathcal{K}$ function

- $\beta(r,s) = r/(ksr+1)$, for any positive constant k, is strictly increasing in r since

$$\frac{\partial \beta}{\partial r} = \frac{1}{(ksr+1)^2} > 0$$

and strictly decreasing in s since

$$\frac{\partial \beta}{\partial s} = \frac{-kr^2}{(ksr+1)^2} < 0$$

$\beta(r,s) \to 0$ as $s \to \infty$. It belongs to class $\mathcal{KL}$

- $\beta(r,s) = r^c e^{-as}$, with positive constants a and c, belongs to class $\mathcal{KL}$

Lemma 4.1

Let α_1 and α_2 be class $\mathcal{K}$ functions on $[0, a_1)$ and $[0, a_2)$, respectively, with $a_1 \geq \lim_{r \to a_2} \alpha_2(r)$, and β be a class $\mathcal{KL}$ function defined on $[0, \lim_{r \to a_2} \alpha_2(r)) \times [0, \infty)$ with $a_1 \geq \lim_{r \to a_2} \beta(\alpha_2(r), 0)$. Let α_3 and α_4 be class $\mathcal{K}_\infty$ functions. Denote the inverse of α_i by α_i^{-1}. Then,

- α_1^{-1} is defined on $[0, \lim_{r \to a_1} \alpha_1(r))$ and belongs to class $\mathcal{K}$
- α_3^{-1} is defined on $[0, \infty)$ and belongs to class $\mathcal{K}_\infty$
- $\alpha_1 \circ \alpha_2$ is defined on $[0, a_2)$ and belongs to class $\mathcal{K}$
- $\alpha_3 \circ \alpha_4$ is defined on $[0, \infty)$ and belongs to class $\mathcal{K}_\infty$
- $\sigma(r, s) = \alpha_1(\beta(\alpha_2(r), s))$ is defined on $[0, a_2) \times [0, \infty)$ and belongs to class $\mathcal{KL}$

Lemma 4.2

Let $V : D \to R$ be a continuous positive definite function defined on a domain $D \subset R^n$ that contains the origin. Let $B_r \subset D$ for some $r > 0$. Then, there exist class $\mathcal{K}$ functions α_1 and α_2, defined on $[0, r]$, such that

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$$

for all $x \in B_r$. If $D = R^n$ and $V(x)$ is radially unbounded, then there exist class $\mathcal{K}_\infty$ functions α_1 and α_2 such that the foregoing inequality holds for all $x \in R^n$

Definition 4.2

The equilibrium point $x = 0$ of $\dot{x} = f(t, x)$ is

- uniformly stable if there exist a class $\mathcal{K}$ function α and a positive constant c, independent of t_0, such that

$$\|x(t)\| \le \alpha(\|x(t_0)\|), \ \forall\ t \ge t_0 \ge 0, \ \forall\ \|x(t_0)\| < c$$

- uniformly asymptotically stable if there exist a class $\mathcal{KL}$ function β and a positive constant c, independent of t_0, such that

$$\|x(t)\| \le \beta(\|x(t_0)\|, t - t_0), \ \forall\ t \ge t_0 \ge 0, \ \forall\ \|x(t_0)\| < c$$

- globally uniformly asymptotically stable if the foregoing inequality is satisfied for any initial state $x(t_0)$

- exponentially stable if there exist positive constants c, k, and λ such that

$$\|x(t)\| \leq k\|x(t_0)\|e^{-\lambda(t-t_0)}, \ \forall \ \|x(t_0)\| < c$$

- globally exponentially stable if the foregoing inequality is satisfied for any initial state $x(t_0)$

Theorem 4.1

Let the origin $x = 0$ be an equilibrium point of $\dot{x} = f(t, x)$ and $D \subset R^n$ be a domain containing $x = 0$. Suppose $f(t, x)$ is piecewise continuous in t and locally Lipschitz in x for all $t \geq 0$ and $x \in D$. Let $V(t, x)$ be a continuously differentiable function such that

$$W_1(x) \leq V(t, x) \leq W_2(x)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq 0$$

for all $t \geq 0$ and $x \in D$, where $W_1(x)$ and $W_2(x)$ are continuous positive definite functions on D. Then, the origin is uniformly stable

Theorem 4.2

Suppose the assumptions of the previous theorem are satisfied with

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \le -W_3(x)$$

for all $t \ge 0$ and $x \in D$, where $W_3(x)$ is a continuous positive definite function on D. Then, the origin is uniformly asymptotically stable. Moreover, if r and c are chosen such that $B_r = \{\|x\| \le r\} \subset D$ and $c < \min_{\|x\|=r} W_1(x)$, then every trajectory starting in $\{W_2(x) \le c\}$ satisfies

$$\|x(t)\| \le \beta(\|x(t_0)\|, t - t_0), \quad \forall \; t \ge t_0 \ge 0$$

for some class $\mathcal{KL}$ function β. Finally, if $D = R^n$ and $W_1(x)$ is radially unbounded, then the origin is globally uniformly asymptotically stable

Theorem 4.3

Suppose the assumptions of the previous theorem are satisfied with

$$k_1\|x\|^a \le V(t,x) \le k_2\|x\|^a$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \le -k_3\|x\|^a$$

for all $t \ge 0$ and $x \in D$, where k_1, k_2, k_3, and a are positive constants. Then, the origin is exponentially stable. If the assumptions hold globally, the origin will be globally exponentially stable

Terminology: A function $V(t,x)$ is said to be

- positive semidefinite if $V(t,x) \geq 0$
- positive definite if $V(t,x) \geq W_1(x)$ for some positive definite function $W_1(x)$
- radially unbounded if $V(t,x) \geq W_1(x)$ and $W_1(x)$ is radially unbounded
- decrescent if $V(t,x) \leq W_2(x)$
- negative definite (semidefinite) if $-V(t,x)$ is positive definite (semidefinite)

Theorems 4.1 and 4.2 say that *the origin is uniformly stable if there is a continuously differentiable, positive definite, decrescent function* $V(t, x)$*, whose derivative along the trajectories of the system is negative semidefinite. It is uniformly asymptotically stable if the derivative is negative definite, and globally uniformly asymptotically stable if the conditions for uniform asymptotic stability hold globally with a radially unbounded* $V(t, x)$

Example 4.2

$$\dot{x} = -[1 + g(t)]x^3, \quad g(t) \geq 0, \ \forall \ t \geq 0$$

$$V(x) = \tfrac{1}{2}x^2$$

$$\dot{V}(t,x) = -[1 + g(t)]x^4 \leq -x^4, \quad \forall \ x \in R, \ \forall \ t \geq 0$$

The origin is globally uniformly asymptotically stable

Example 4.3

$$\dot{x}_1 = -x_1 - g(t)x_2, \qquad \dot{x}_2 = x_1 - x_2$$

$$0 \leq g(t) \leq k \ \text{ and } \ \dot{g}(t) \leq g(t), \ \forall \ t \geq 0$$

$$V(t,x) = x_1^2 + [1 + g(t)]x_2^2$$

$$x_1^2 + x_2^2 \le V(t,x) \le x_1^2 + (1+k)x_2^2, \quad \forall\ x \in R^2$$

$$\dot{V}(t,x) = -2x_1^2 + 2x_1x_2 - [2 + 2g(t) - \dot{g}(t)]x_2^2$$

$$2 + 2g(t) - \dot{g}(t) \ge 2 + 2g(t) - g(t) \ge 2$$

$$\dot{V}(t,x) \le -2x_1^2 + 2x_1x_2 - 2x_2^2 = -\ x^T \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} x$$

The origin is globally exponentially stable