Nonlinear Control

Lecture # 1

Introduction

&

Two-Dimensional Systems

Nonlinear State Model

$$
\begin{aligned}
\dot{x}_1 &= f_1(t, x_1, \ldots, x_n, u_1, \ldots, u_m) \\
\dot{x}_2 &= f_2(t, x_1, \ldots, x_n, u_1, \ldots, u_m) \\
\vdots \quad & \quad \vdots \\
\dot{x}_n &= f_n(t, x_1, \ldots, x_n, u_1, \ldots, u_m)
\end{aligned}
$$

$\dot{x}_i$ denotes the derivative of x_i with respect to the time variable t

u_1, u_2, ..., u_m are input variables

x_1, x_2, ..., x_n the state variables

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}, \quad f(t,x,u) = \begin{bmatrix} f_1(t,x,u) \\ f_2(t,x,u) \\ \vdots \\ \vdots \\ f_n(t,x,u) \end{bmatrix}$$

$$\dot{x} = f(t,x,u)$$

$$
\begin{aligned}
\dot{x} &= f(t, x, u) \\
y &= h(t, x, u)
\end{aligned}
$$

x is the state, u is the input
y is the output (q-dimensional vector)

Special Cases:
Linear systems:

$$
\begin{aligned}
\dot{x} &= A(t)x + B(t)u \\
y &= C(t)x + D(t)u
\end{aligned}
$$

Unforced state equation:

$$\dot{x} = f(t, x)$$

Results from $\dot{x} = f(t, x, u)$ with $u = \gamma(t, x)$

Autonomous System:

$$\dot{x} = f(x)$$

Time-Invariant System:

$$\dot{x} = f(x, u)$$
$$y = h(x, u)$$

A time-invariant state model has a time-invariance property with respect to shifting the initial time from t_0 to $t_0 + a$, provided the input waveform is applied from $t_0 + a$ rather than t_0

Existence and Uniqueness of Solutions

$$\dot{x} = f(t,x)$$

$f(t,x)$ is piecewise continuous in t and locally Lipschitz in x over the domain of interest

$f(t,x)$ is piecewise continuous in t on an interval $J \subset R$ if for every bounded subinterval $J_0 \subset J$, f is continuous in t for all $t \in J_0$, except, possibly, at a finite number of points where f may have finite-jump discontinuities

$f(t,x)$ is locally Lipschitz in x at a point x_0 if there is a neighborhood $N(x_0,r) = \{x \in R^n \mid \|x - x_0\| < r\}$ where $f(t,x)$ satisfies the Lipschitz condition

$$\|f(t,x) - f(t,y)\| \ \leq \ L\|x - y\|, \quad L > 0$$

A function $f(t, x)$ is locally Lipschitz in x on a domain (open and connected set) $D \subset R^n$ if it is locally Lipschitz at every point $x_0 \in D$

When $n = 1$ and f depends only on x

$$\frac{|f(y) - f(x)|}{|y - x|} \leq L$$

On a plot of $f(x)$ versus x, a straight line joining any two points of $f(x)$ cannot have a slope whose absolute value is greater than L

Any function $f(x)$ that has infinite slope at some point is not locally Lipschitz at that point

Lemma 1.1

Let $f(t, x)$ be piecewise continuous in t and locally Lipschitz in x at x_0, for all $t \in [t_0, t_1]$. Then, there is $\delta > 0$ such that the state equation $\dot{x} = f(t, x)$, with $x(t_0) = x_0$, has a unique solution over $[t_0, t_0 + \delta]$

Without the local Lipschitz condition, we cannot ensure uniqueness of the solution. For example, $\dot{x} = x^{1/3}$ has $x(t) = (2t/3)^{3/2}$ and $x(t) \equiv 0$ as two different solutions when the initial state is $x(0) = 0$

The lemma is a local result because it guarantees existence and uniqueness of the solution over an interval $[t_0, t_0 + \delta]$, but this interval might not include a given interval $[t_0, t_1]$. Indeed the solution may cease to exist after some time

Example 1.3

$$\dot{x} = -x^2$$

$f(x) = -x^2$ is locally Lipschitz for all x

$$x(0) = -1 \quad \Rightarrow \quad x(t) = \frac{1}{(t-1)}$$

$$x(t) \to -\infty \;\; \text{as} \;\; t \to 1$$

The solution has a *finite escape time* at $t = 1$

In general, if $f(t, x)$ is locally Lipschitz over a domain D and the solution of $\dot{x} = f(t, x)$ has a finite escape time t_e, then the solution $x(t)$ must leave every compact (closed and bounded) subset of D as $t \to t_e$

Global Existence and Uniqueness

A function $f(t,x)$ is globally Lipschitz in x if

$$\|f(t,x) - f(t,y)\| \;\leq\; L\|x - y\|$$

for all $x, y \in R^n$ with the same Lipschitz constant L

If $f(t,x)$ and its partial derivatives $\partial f_i/\partial x_j$ are continuous for all $x \in R^n$, then $f(t,x)$ is globally Lipschitz in x if and only if the partial derivatives $\partial f_i/\partial x_j$ are globally bounded, uniformly in t

$f(x) = -x^2$ is locally Lipschitz for all x but not globally Lipschitz because $f'(x) = -2x$ is not globally bounded

Lemma 1.2

Let $f(t, x)$ be piecewise continuous in t and globally Lipschitz in x for all $t \in [t_0, t_1]$. Then, the state equation $\dot{x} = f(t, x)$, with $x(t_0) = x_0$, has a unique solution over $[t_0, t_1]$

The global Lipschitz condition is satisfied for linear systems of the form

$$\dot{x} = A(t)x + g(t)$$

but it is a restrictive condition for general nonlinear systems

Lemma 1.3

Let $f(t,x)$ be piecewise continuous in t and locally Lipschitz in x for all $t \geq t_0$ and all x in a domain $D \subset R^n$. Let W be a compact subset of D, and suppose that every solution of

$$\dot{x} = f(t,x), \quad x(t_0) = x_0$$

with $x_0 \in W$ lies entirely in W. Then, there is a unique solution that is defined for all $t \geq t_0$

Change of Variables

$$\text{Map: } z = T(x), \qquad \text{Inverse map: } x = T^{-1}(z)$$

Definitions

- a map $T(x)$ is invertible over its domain D if there is a map $T^{-1}(\cdot)$ such that $x = T^{-1}(z)$ for all $z \in T(D)$
- A map $T(x)$ is a diffeomorphism if $T(x)$ and $T^{-1}(x)$ are continuously differentiable
- $T(x)$ is a local diffeomorphism at x_0 if there is a neighborhood N of x_0 such that T restricted to N is a diffeomorphism on N
- $T(x)$ is a global diffeomorphism if it is a diffeomorphism on R^n and $T(R^n) = R^n$

Jacobian matrix

$$\frac{\partial T}{\partial x} = \begin{bmatrix} \frac{\partial T_1}{\partial x_1} & \frac{\partial T_1}{\partial x_2} & \cdots & \frac{\partial T_1}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial T_n}{\partial x_1} & \frac{\partial T_n}{\partial x_2} & \cdots & \frac{\partial T_n}{\partial x_n} \end{bmatrix}$$

Lemma 1.4

The continuously differentiable map $z = T(x)$ is a local diffeomorphism at x_0 if the Jacobian matrix $[\partial T/\partial x]$ is nonsingular at x_0. It is a global diffeomorphism if and only if $[\partial T/\partial x]$ is nonsingular for all $x \in R^n$ and T is proper; that is, $\lim_{\|x\|\to\infty} \|T(x)\| = \infty$

Equilibrium Points

A point $x = x^*$ in the state space is said to be an equilibrium point of $\dot{x} = f(t, x)$ if

$$x(t_0) = x^* \quad \Rightarrow \quad x(t) \equiv x^*, \;\; \forall \; t \geq t_0$$

For the autonomous system $\dot{x} = f(x)$, the equilibrium points are the real solutions of the equation

$$f(x) = 0$$

An equilibrium point could be isolated; that is, there are no other equilibrium points in its vicinity, or there could be a continuum of equilibrium points

Two-Dimensional Systems

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) &= f_1(x) \\ \dot{x}_2 &= f_2(x_1, x_2) &= f_2(x) \end{aligned}$$

Let $x(t) = (x_1(t), x_2(t))$ be a solution that starts at initial state $x_0 = (x_{10}, x_{20})$. The locus in the x_1–x_2 plane of the solution $x(t)$ for all $t \geq 0$ is a curve that passes through the point x_0. This curve is called a *trajectory* or *orbit*

The x_1–x_2 plane is called the *state plane* or *phase plane*

The family of all trajectories is called the *phase portrait*

The *vector field* $f(x) = (f_1(x), f_2(x))$ is tangent to the trajectory at point x because

$$\frac{dx_2}{dx_1} = \frac{f_2(x)}{f_1(x)}$$

Qualitative Behavior of Linear Systems

$$\dot{x} = Ax, \qquad A \text{ is a } 2 \times 2 \text{ real matrix}$$

$$x(t) = M \exp(J_r t) M^{-1} x_0$$

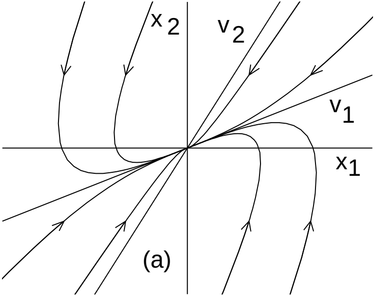
When A has distinct eigenvalues,

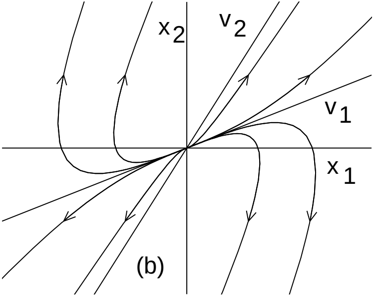
$$J_r = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

$$x(t) = Mz(t) \;\Rightarrow\; \dot{z} = J_r z(t)$$

Case 1. Both eigenvalues are real:

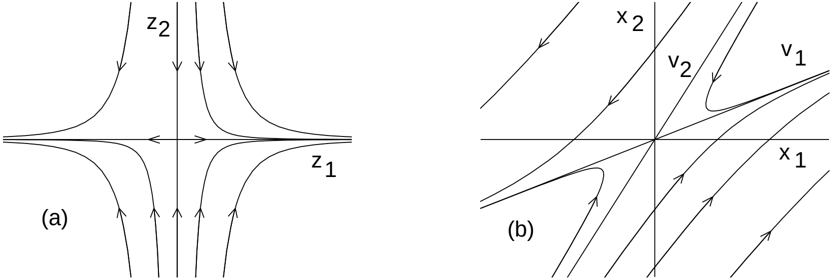
$$M = [v_1, v_2]$$





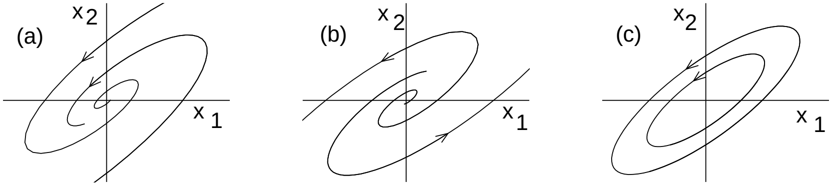
Stable Node: $\lambda_2 < \lambda_1 < 0$

Unstable Node: $\lambda_2 > \lambda_1 > 0$



Saddle: $\lambda_2 < 0 < \lambda_1$

Case 2. Complex eigenvalues: $\lambda_{1,2} = \alpha \pm j\beta$



$\alpha < 0$	$\alpha > 0$	$\alpha = 0$
Stable Focus	Unstable Focus	Center

Qualitative Behavior Near Equilibrium Points

Let $p = (p_1, p_2)$ be an equilibrium point of the system

$$\dot{x}_1 = f_1(x_1, x_2), \qquad \dot{x}_2 = f_2(x_1, x_2)$$

where f_1 and f_2 are continuously differentiable

Expand f_1 and f_2 in Taylor series about (p_1, p_2)

$$\begin{aligned}
\dot{x}_1 &= f_1(p_1, p_2) + a_{11}(x_1 - p_1) + a_{12}(x_2 - p_2) + \text{H.O.T.} \\
\dot{x}_2 &= f_2(p_1, p_2) + a_{21}(x_1 - p_1) + a_{22}(x_2 - p_2) + \text{H.O.T.}
\end{aligned}$$

$$a_{11} = \left.\frac{\partial f_1(x_1, x_2)}{\partial x_1}\right|_{x=p}, \qquad a_{12} = \left.\frac{\partial f_1(x_1, x_2)}{\partial x_2}\right|_{x=p}$$

$$a_{21} = \left.\frac{\partial f_2(x_1, x_2)}{\partial x_1}\right|_{x=p}, \qquad a_{22} = \left.\frac{\partial f_2(x_1, x_2)}{\partial x_2}\right|_{x=p}$$

$$f_1(p_1,p_2) = f_2(p_1,p_2) = 0$$

$$y_1 = x_1 - p_1 \qquad y_2 = x_2 - p_2$$

$$\begin{aligned}
\dot{y}_1 &= \dot{x}_1 = a_{11}y_1 + a_{12}y_2 + \text{H.O.T.} \\
\dot{y}_2 &= \dot{x}_2 = a_{21}y_1 + a_{22}y_2 + \text{H.O.T.}
\end{aligned}$$

$$\dot{y} \approx Ay$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}\right|_{x=p} = \left.\frac{\partial f}{\partial x}\right|_{x=p}$$

Eigenvalues of A	Type of equilibrium point of the nonlinear system
$\lambda_2 < \lambda_1 < 0$	Stable Node
$\lambda_2 > \lambda_1 > 0$	Unstable Node
$\lambda_2 < 0 < \lambda_1$	Saddle
$\alpha \pm j\beta,\ \alpha < 0$	Stable Focus
$\alpha \pm j\beta,\ \alpha > 0$	Unstable Focus
$\pm j\beta$	Linearization Fails

Example 2.1

$$\begin{aligned} \dot{x}_1 &= -x_2 - \mu x_1(x_1^2 + x_2^2) \\ \dot{x}_2 &= x_1 - \mu x_2(x_1^2 + x_2^2) \end{aligned}$$

$x = 0$ is an equilibrium point

$$\frac{\partial f}{\partial x} = \left[\begin{array}{cc} -\mu(3x_1^2 + x_2^2) & -(1 + 2\mu x_1 x_2) \\ (1 - 2\mu x_1 x_2) & -\mu(x_1^2 + 3x_2^2) \end{array} \right]$$

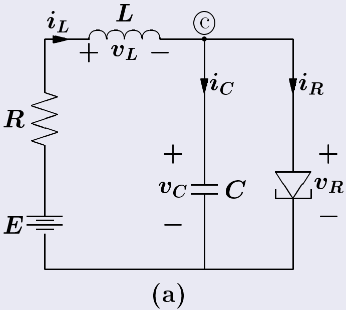
$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \left[\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right]$$

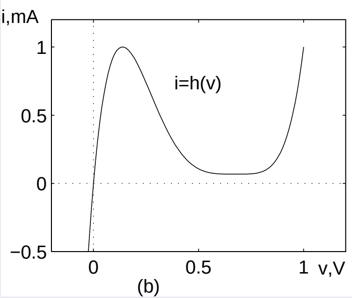
$$x_1 = r\cos\theta \text{ and } x_2 = r\sin\theta \;\Rightarrow\; \dot{r} = -\mu r^3 \text{ and } \dot{\theta} = 1$$

Stable focus when $\mu > 0$ and Unstable focus when $\mu < 0$

Multiple Equilibria

Example 2.2: Tunnel-diode circuit

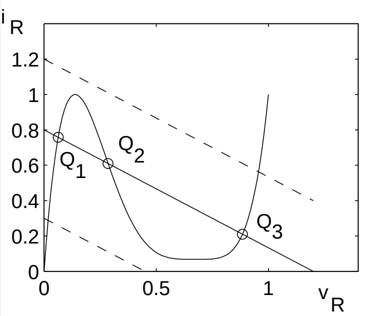




$$x_1 = v_C, \quad x_2 = i_L$$

$$
\begin{aligned}
\dot{x}_1 &= 0.5[-h(x_1) + x_2] \\
\dot{x}_2 &= 0.2(-x_1 - 1.5x_2 + 1.2)
\end{aligned}
$$

$$h(x_1) = 17.76x_1 - 103.79x_1^2 + 229.62x_1^3 - 226.31x_1^4 + 83.72x_1^5$$



$$
\begin{aligned}
Q_1 &= (0.063, 0.758) \\
Q_2 &= (0.285, 0.61) \\
Q_3 &= (0.884, 0.21)
\end{aligned}
$$

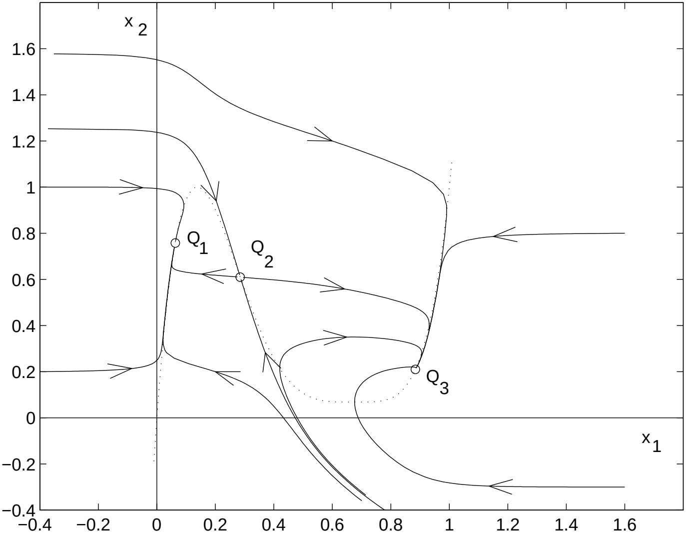
$$\frac{\partial f}{\partial x} = \begin{bmatrix} -0.5h'(x_1) & 0.5 \\ -0.2 & -0.3 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} -3.598 & 0.5 \\ -0.2 & -0.3 \end{bmatrix}, \quad \text{Eigenvalues : } -3.57, \ -0.33$$

$$A_2 = \begin{bmatrix} 1.82 & 0.5 \\ -0.2 & -0.3 \end{bmatrix}, \quad \text{Eigenvalues : } 1.77, \ -0.25$$

$$A_3 = \begin{bmatrix} -1.427 & 0.5 \\ -0.2 & -0.3 \end{bmatrix}, \quad \text{Eigenvalues : } -1.33, \ -0.4$$

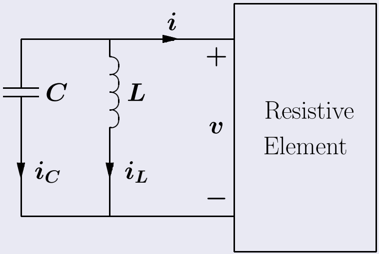
Q_1 is a stable node; Q_2 is a saddle; Q_3 is a stable node



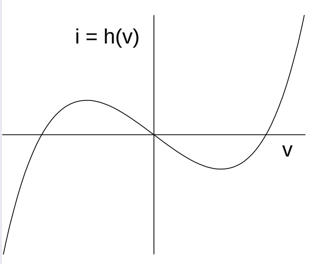
x 2
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
x 1
Q 1
Q 2
Q 3

Limit Cycles

Example: Negative Resistance Oscillator



(a)



(b)

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 - \varepsilon h'(x_1)x_2
\end{aligned}
$$

There is a unique equilibrium point at the origin

$$
A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} 0 & 1 \\ -1 & -\varepsilon h'(0) \end{bmatrix}
$$

$$
\lambda^2 + \varepsilon h'(0)\lambda + 1 = 0
$$

$$
h'(0) < 0 \;\Rightarrow\; \text{Unstable Focus or Unstable Node}
$$

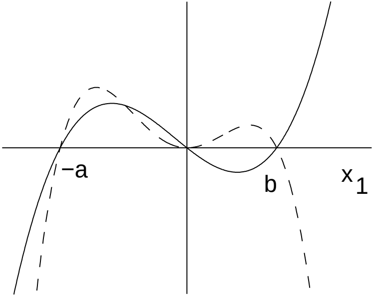
Energy Analysis:

$$E = \tfrac{1}{2}Cv_C^2 + \tfrac{1}{2}Li_L^2$$

$$v_C = x_1 \quad \text{and} \quad i_L = -h(x_1) - \frac{1}{\varepsilon}x_2$$

$$E = \tfrac{1}{2}C\{x_1^2 + [\varepsilon h(x_1) + x_2]^2\}$$

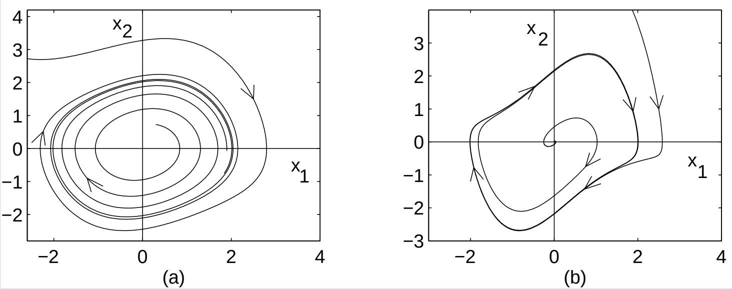
$$\begin{aligned}
\dot{E} &= C\{x_1\dot{x}_1 + [\varepsilon h(x_1) + x_2][\varepsilon h'(x_1)\dot{x}_1 + \dot{x}_2]\} \\
&= C\{x_1x_2 + [\varepsilon h(x_1) + x_2][\varepsilon h'(x_1)x_2 - x_1 - \varepsilon h'(x_1)x_2]\} \\
&= C[x_1x_2 - \varepsilon x_1 h(x_1) - x_1x_2] \\
&= -\varepsilon C x_1 h(x_1)
\end{aligned}$$



$$\dot{E} = -\varepsilon C x_1 h(x_1)$$

Example 2.4: Van der Pol Oscillator

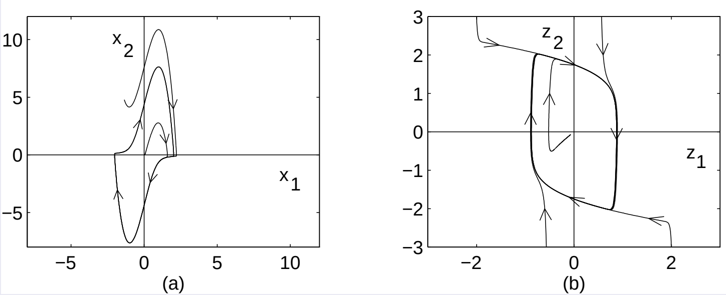
$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 + \varepsilon(1 - x_1^2)x_2
\end{aligned}
$$



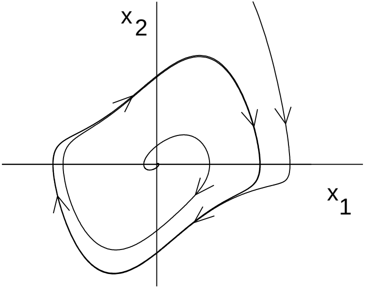
$\varepsilon = 0.2$ $\varepsilon = 1$

$$\dot{z}_1 = \frac{1}{\varepsilon} z_2$$

$$\dot{z}_2 = -\varepsilon(z_1 - z_2 + \tfrac{1}{3} z_2^3)$$

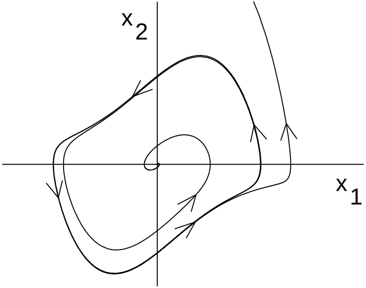


$\varepsilon = 5$



(a)

Stable Limit Cycle



(b)

Unstable Limit Cycle