Nonlinear Control

Lecture # 10

State Feedback Stabilization

and

Robust State Feedback Stabilization

Passivity-Based Control

$$\dot{x} = f(x, u), \qquad y = h(x), \qquad f(0, 0) = 0$$

$$u^T y \geq \dot{V} = \frac{\partial V}{\partial x} f(x, u)$$

Theorem 9.1

If the system is

(1) passive with a radially unbounded positive definite storage function and

(2) zero-state observable,

then the origin can be globally stabilized by

$$u = -\phi(y), \qquad \phi(0) = 0, \quad y^T \phi(y) > 0 \ \forall \ y \neq 0$$

Proof

$$\dot{V} = \frac{\partial V}{\partial x} f(x, -\phi(y)) \leq -y^T \phi(y) \leq 0$$

$$\dot{V}(x(t)) \equiv 0 \;\Rightarrow\; y(t) \equiv 0 \;\Rightarrow\; u(t) \equiv 0 \;\Rightarrow\; x(t) \equiv 0$$

Apply the invariance principle

A given system may be made passive by

(1) Choice of output,
(2) Feedback,

or both

Choice of Output

$$\dot{x} = f(x) + G(x)u, \qquad \frac{\partial V}{\partial x} f(x) \le 0, \quad \forall\ x$$

No output is defined. Choose the output as

$$y = h(x) \stackrel{\text{def}}{=} \left[\frac{\partial V}{\partial x} G(x) \right]^T$$

$$\dot{V} = \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} G(x) u \le y^T u$$

Check zero-state observability

Example 9.14

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1^3 + u$$

$$V(x) = \tfrac{1}{4}x_1^4 + \tfrac{1}{2}x_2^2$$

$$\text{With } u = 0 \qquad \dot{V} = x_1^3 x_2 - x_2 x_1^3 = 0$$

$$\text{Take } y = \frac{\partial V}{\partial x} G = \frac{\partial V}{\partial x_2} = x_2$$

Is it zero-state observable?

$$\text{with } u = 0, \qquad y(t) \equiv 0 \;\;\Rightarrow\;\; x(t) \equiv 0$$

$$u = -kx_2 \qquad \text{or} \qquad u = -(2k/\pi)\tan^{-1}(x_2) \qquad (k > 0)$$

Feedback Passivation

Definition

The system

$$\dot{x} = f(x) + G(x)u, \quad y = h(x) \quad (*)$$

is equivalent to a passive system if $\exists\ u = \alpha(x) + \beta(x)v$ such that

$$\dot{x} = f(x) + G(x)\alpha(x) + G(x)\beta(x)v, \qquad y = h(x)$$

is passive

Theorem [20]

The system (*) is locally equivalent to a passive system (with a positive definite storage function) if it has relative degree one at $x = 0$ and the zero dynamics have a stable equilibrium point at the origin with a positive definite Lyapunov function

Example 9.15 (m-link Robot Manipulator)

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + D\dot{q} + g(q) = u$$

$$M = M^T > 0,\ (\dot{M} - 2C)^T = -(\dot{M} - 2C),\ D = D^T \geq 0$$

Stabilize the system at $q = q_r$

$$e = q - q_r, \qquad \dot{e} = \dot{q}$$

$$M(q)\ddot{e} + C(q,\dot{q})\dot{e} + D\dot{e} + g(q) = u$$

$(e = 0, \dot{e} = 0)$ is not an open-loop equilibrium point

$$u = g(q) - K_p e + v, \qquad (K_p = K_p^T > 0)$$

$$M(q)\ddot{e} + C(q,\dot{q})\dot{e} + D\dot{e} + K_p e = v$$

$$M(q)\ddot{e} + C(q,\dot{q})\dot{e} + D\dot{e} + K_p e = v$$

$$V = \tfrac{1}{2}\dot{e}^T M(q)\dot{e} + \tfrac{1}{2}e^T K_p e$$

$$\dot{V} = \tfrac{1}{2}\dot{e}^T(\dot{M} - 2C)\dot{e} - \dot{e}^T D\dot{e} - \dot{e}^T K_p e + \dot{e}^T v + e^T K_p \dot{e} \le \dot{e}^T v$$

$$y = \dot{e}$$

Is it zero-state observable? Set $v = 0$

$$\dot{e}(t) \equiv 0 \;\Rightarrow\; \ddot{e}(t) \equiv 0 \;\Rightarrow\; K_p e(t) \equiv 0 \;\Rightarrow\; e(t) \equiv 0$$

$$v = -\phi(\dot{e}), \qquad [\phi(0) = 0,\; \dot{e}^T\phi(\dot{e}) > 0, \;\; \forall \dot{e} \neq 0]$$

$$u = g(q) - K_p e - \phi(\dot{e})$$

Special case: $\quad u = g(q) - K_p e - K_d\dot{e}, \qquad K_d = K_d^T > 0$

Passivity-Based Control: Cascade Connection

$$\dot{x} = f_a(x) + F(x, y)y, \quad \dot{z} = f(z) + G(z)u, \quad y = h(z)$$

$$f_a(0) = 0, \quad f(0) = 0, \quad h(0) = 0$$

$$\frac{\partial V}{\partial z} f(z) + \frac{\partial V}{\partial z} G(z)u \le y^T u, \quad \frac{\partial W}{\partial x} f_a(x) \le 0$$

$$U(x, z) = W(x) + V(z)$$

$$\dot{U} \le \frac{\partial W}{\partial x} F(x, y)y + y^T u \;=\; y^T \left[u + \left(\frac{\partial W}{\partial x} F(x, y) \right)^T \right]$$

$$u = -\left(\frac{\partial W}{\partial x} F(x, y) \right)^T + v \;\Rightarrow\; \dot{U} \le y^T v$$

The system

$$
\begin{aligned}
\dot{x} &= f_a(x) + F(x,y)y \\
\dot{z} &= f(z) - G(z)\left(\frac{\partial W}{\partial x}F(x,y)\right)^T + G(z)v \\
y &= h(z)
\end{aligned}
$$

with input v and output y is passive with storage function U

$$v = -\phi(y), \quad [\phi(0) = 0,\; y^T\phi(y) > 0 \;\forall\; y \neq 0]$$

$$\dot{U} \leq \frac{\partial W}{\partial x} f_a(x) - y^T\phi(y) \leq 0, \dot{U} = 0 \Rightarrow x = 0 \& y = 0 \Rightarrow u = 0$$

ZSO of driving system: $\dot{U}(t) \equiv 0 \;\Rightarrow\; z(t) \equiv 0$

Theorem 9.2

Suppose

- the system

$$\dot{z} = f(z) + G(z)u, \qquad y = h(z)$$

  is zero-state observable and passive with a radially unbounded, positive definite storage function;
- the origin of $\dot{x} = f_a(x)$ is globally asymptotically stable and $W(x)$ is a radially unbounded, positive definite Lyapunov function

Then, $$u = -\left(\frac{\partial W}{\partial x} F(x,y)\right)^T - \phi(y),$$

globally stabilizes the origin $(x = 0, z = 0)$

Example 9.16 (see Examples 9.7 and 9.12)

$$\dot{x} = -x + x^2 z, \qquad \dot{z} = u$$

With $y = z$ as the output, the system takes the form of the cascade connection

$$\dot{z} = u, \quad y = z$$

is passive with $V(z) = \frac{1}{2}z^2$ and zero-state observable

$$\dot{x} = -x, \quad W(x) = \frac{1}{2}x^2 \;\Rightarrow\; \dot{W} = -x^2$$

$$u = -x^3 - kz, \quad k > 0$$

Robust State Feedback Stabilization

Definition 10.1

The system

$$\dot{x} = f(x,u) + \delta(t,x,u)$$

is said to be practically stabilizable by the feedback control $u = \phi(x)$ if for every $b > 0$, the control can be designed such that the solutions are ultimately bounded by b; that is

$$\|x(t)\| \le b, \qquad \forall\, t \ge T, \quad \text{for some } T > 0$$

- local practical stabilization
- regional practical stabilization
- global practical stabilization
- semiglobal practical stabilization

Sliding Mode Control

Example 10.1

$$\dot{x}_1 = x_2 \qquad \dot{x}_2 = h(x) + g(x)u, \qquad g(x) \geq g_0 > 0$$

Sliding Manifold (Surface):

$$s = ax_1 + x_2 = 0$$

$$s(t) \equiv 0 \quad \Rightarrow \quad \dot{x}_1 = -ax_1$$

$$a > 0 \quad \Rightarrow \quad \lim_{t \to \infty} x_1(t) = 0$$

How can we bring the trajectory to the manifold $s = 0$?

How can we maintain it there?

$$\dot{s} = a\dot{x}_1 + \dot{x}_2 = ax_2 + h(x) + g(x)u$$

Suppose

$$\left| \frac{ax_2 + h(x)}{g(x)} \right| \leq \varrho(x)$$

$$V = \tfrac{1}{2}s^2$$

$$\dot{V} = s\dot{s} = s[ax_2 + h(x)] + g(x)su \leq g(x)|s|\varrho(x) + g(x)su$$

$$\beta(x) \geq \varrho(x) + \beta_0, \quad \beta_0 > 0$$

$$s > 0, \quad u = -\beta(x)$$

$$\dot{V} \leq g(x)|s|\varrho(x) - g(x)\beta(x)|s| \leq -g(x)\beta_0|s|$$

$$s < 0, \qquad u = \beta(x)$$

$$\dot{V} \le g(x)|s|\varrho(x) + g(x)su = g(x)|s|\varrho(x) - g(x)\beta(x)|s|$$

$$\dot{V} \le g(x)|s|\varrho(x) - g(x)(\varrho(x) + \beta_0)|s| = -g(x)\beta_0|s|$$

$$\mathrm{sgn}(s) = \begin{cases} 1, & s > 0 \\ -1, & s < 0 \end{cases}$$

$$u = -\beta(x)\ \mathrm{sgn}(s) \ \Rightarrow\ \dot{V} \le -g(x)\beta_0|s| \le -g_0\beta_0|s|$$

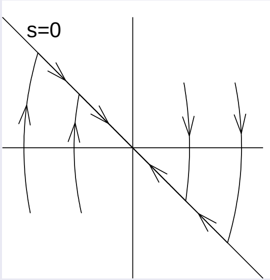
$$\dot{V} \le -g_0\beta_0\sqrt{2V}$$

$$\frac{dV}{\sqrt{V}} \le -g_0\beta_0\sqrt{2}\ dt \ \Rightarrow\ \sqrt{V(s(t))} \le \sqrt{V(s(0))} - g_0\beta_0\frac{1}{\sqrt{2}}\ t$$

$$|s(t)| \leq |s(0)| - g_0\beta_0\ t$$

$s(t)$ reaches zero in finite time

The trajectory cannot leave the surface $s = 0$



Pendulum Equation

$$\ddot{\theta} + \sin\theta + b\dot{\theta} = cu$$

$$0 \le b \le 0.2, \quad 0.5 \le c \le 2$$

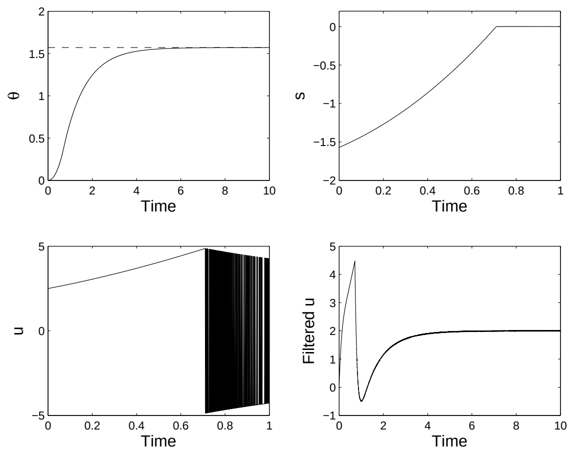
Stabilize the pendulum at $\theta = \pi/2$

$$x_1 = \theta - \frac{\pi}{2},\ x_2 = \dot{\theta} \ \ \Rightarrow \ \ \dot{x}_1 = x_2, \ \ \dot{x}_2 = -\cos x_1 - bx_2 + cu$$

$$s = x_1 + x_2, \quad \dot{s} = x_2 - \cos x_1 - bx_2 + cu$$

$$\left|\frac{x_2 - \cos x_1 - bx_2}{c}\right| = \left|\frac{(1-b)x_2 - \cos x_1}{c}\right| \le 2(|x_2| + 1)$$

$$u = -(2.5 + 2|x_2|)\ \mathrm{sgn}(s)$$



$$b = 0.01, \ c = 0.5, \ \theta(0) = \dot{\theta}(0) = 0$$

Estimate the region of attraction when

$$\left|\frac{ax_2 + h(x)}{g(x)}\right| \le \varrho(x), \quad \forall x \in D \subset R^2$$

$$\dot{x}_1 = -ax_1 + s \quad \dot{s} = ax_2 + h(x) - g(x)\beta(x)\mathrm{sgn}(s)$$

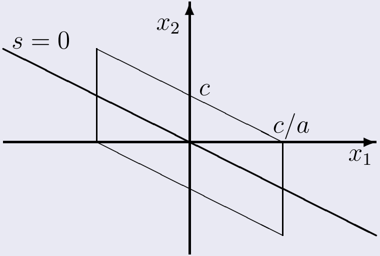
$$s\dot{s} \le -g_0\beta_0|s| \;\Rightarrow\; \{|s| \le c\} \text{ is positively invariant}$$

$$V_0 = \tfrac{1}{2}x_1^2 \;\Rightarrow\; \dot{V}_0 = x_1\dot{x}_1 = -ax_1^2 + x_1 s \le -ax_1^2 + |x_1|c$$

$$\dot{V}_0 \le 0, \quad \forall\; |s| \le c \text{ and } |x_1| \ge \frac{c}{a}$$

$$\Omega = \left\{|x_1| \le \frac{c}{a},\; |s| \le c\right\} \subset D \text{ is positively invariant}$$

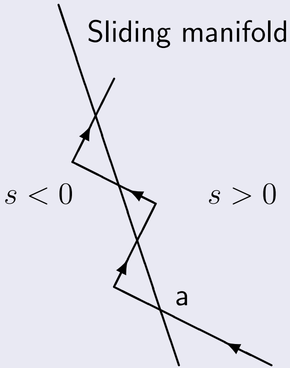
$$\Omega = \left\{ |x_1| \leq \frac{c}{a}, \ |s| \leq c \right\}$$



$$\left| \frac{a x_2 + h(x)}{g(x)} \right| \leq k_1 < k, \quad \forall \ x \in \Omega$$

$$u = -k \ \mathrm{sgn}(s)$$

Chattering



How can we reduce or eliminate chattering?

Reduce the amplitude of the signum function

$$\dot{s} = ax_2 + h(x) + g(x)u$$

$$u = -\frac{[ax_2 + \hat{h}(x)]}{\hat{g}(x)} + v$$

$$\dot{s} = \delta(x) + g(x)v$$

$$\delta(x) = a\left[1 - \frac{g(x)}{\hat{g}(x)}\right] x_2 + h(x) - \frac{g(x)}{\hat{g}(x)}\hat{h}(x)$$

$$\left|\frac{\delta(x)}{g(x)}\right| \le \varrho(x), \qquad \beta(x) \ge \varrho(x) + \beta_0$$

$$v = -\beta(x)\,\mathrm{sgn}(s)$$

Back to the pendulum equation

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\cos x_1 - bx_2 + cu, \qquad \hat{b} = 0, \hat{c}$$

$$u = \frac{-x_2 + \cos x_1}{\hat{c}} + v \;\Rightarrow\; \dot{s} = \delta + cv$$

$$\frac{\delta}{c} = \left(\frac{1-b}{c} - \frac{1}{\hat{c}}\right) x_2 - \left(\frac{1}{c} - \frac{1}{\hat{c}}\right) \cos x_1$$

Take $\hat{c} = 1/1.2$ to minimize $|(1-b)/c - 1/\hat{c}|$

$$\left|\frac{\delta}{c}\right| \le 0.8|x_2| + 0.8$$

$$u = 1.2\cos x_1 - 1.2x_2 - (1 + 0.8|x_2|)\,\mathrm{sgn}(s)$$

Simulation with unmodeled actuator dynamics

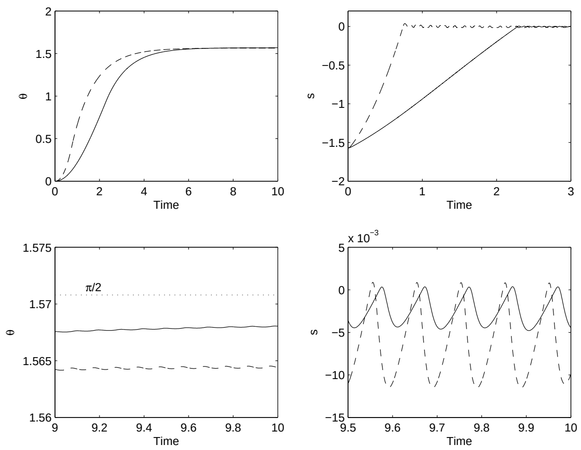
$$\frac{1}{(0.01s+1)^2}$$

Dashed lines:

$$u = -(2.5 + 2|x_2|)\ \mathrm{sgn}(s)$$

Solid lines:

$$u = 1.2\cos x_1 - 1.2x_2 - (1 + 0.8|x_2|)\ \mathrm{sgn}(s)$$

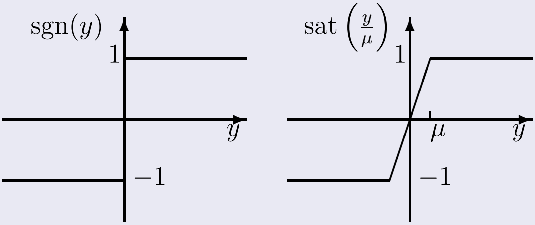


$$b = 0.01,\ c = 0.5,\ \theta(0) = \dot{\theta}(0) = 0$$

Replace the signum function by a high-slope saturation function

$$u = -\beta(x)\ \mathrm{sat}\left(\frac{s}{\mu}\right)$$

$$\mathrm{sat}(y) = \begin{cases} y, & \text{if } |y| \le 1 \\ \mathrm{sgn}(y), & \text{if } |y| > 1 \end{cases}$$



How can we analyze the system?

$$\text{For } |s| \geq \mu, \quad u = -\beta(x)\,\mathrm{sgn}(s)$$

With $c \geq \mu$

- $\Omega = \left\{|x_1| \leq \frac{c}{a},\ |s| \leq c\right\}$ is positively invariant
- The trajectory reaches the boundary layer $\{|s| \leq \mu\}$in finite time
- The boundary layer is positively invariant

Inside the boundary layer:

$$\dot{x}_1 = -ax_1 + s \qquad \dot{s} = ax_2 + h(x) - g(x)\beta(x)\frac{s}{\mu}$$

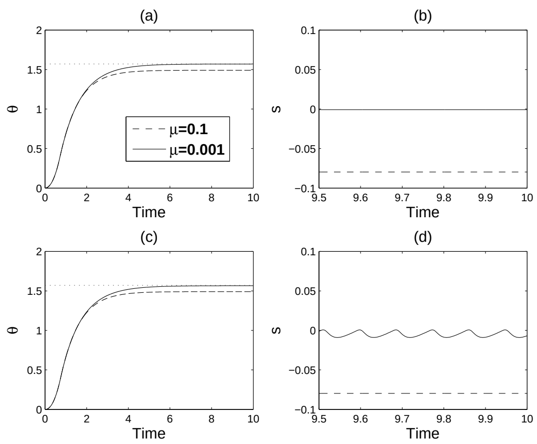
$$x_1\dot{x}_1 \le -ax_1^2 + |x_1|\mu$$

$$x_1\dot{x}_1 \le -(1-\theta_1)ax_1^2, \quad \forall\ |x_1| \ge \frac{\mu}{\theta_1 a}, \quad 0 < \theta_1 < 1$$

The trajectories reach the positively invariant set

$$\Omega_\mu = \{|x_1| \le \frac{\mu}{\theta_1 a},\ |s| \le \mu\}$$

in finite time



(a) & (b) without and (c) & (d) with actuator dynamics

Special case $h(0) = 0$

Inside the boundary layer, the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = h(x) - [g(x)\beta(x)/\mu]\,(ax_1 + x_2)$$

has an equilibrium point at the origin. We can stabilize the origin by choosing μ small enough

Pendulum equation: Stabilize the pendulum at $\theta = \pi$

$$x_1 = \theta - \pi, \quad x_2 = \dot{\theta}, \quad s = x_1 + x_2$$

$$\dot{s} = x_2 + \sin x_1 - bx_2 + cu$$

$$0 \le b \le 0.2, \;\; 0.5 \le c \le 2$$

$$\left| \frac{(1-b)x_2 + \sin x_1}{c} \right| \le 2(|x_1| + |x_2|)$$

$$u = -2(|x_1| + |x_2| + 1)\ \mathrm{sat}(s/\mu)$$

Inside the boundary layer

$$\dot{x}_1 = -x_1 + s, \quad \dot{s} = (1-b)x_2 + \sin x_1 - 2c(|x_1| + |x_2| + 1)s/\mu$$

$$V_1 = \frac{1}{2}x_1^2 + \frac{1}{2}s^2$$

$$\dot{V}_1 \le - \begin{bmatrix} |x_1| \\ |s| \end{bmatrix}^T \begin{bmatrix} 1 & -3/2 \\ -3/2 & (1/\mu \; - \; 1) \end{bmatrix} \begin{bmatrix} |x_1| \\ |s| \end{bmatrix}$$