Nonlinear Control

Lecture # 11

Robust State Feedback Stabilization

Sliding Mode Control

$$\dot{x} = f(x) + B(x)[G(x)u + \delta(t, x, u)]$$

$x \in R^n$, $u \in R^m$, f and B are known, while G and δ could be uncertain, $f(0) = 0$, $G(x)$ is a positive definite symmetric matrix with

$$\lambda_{min}(G(x)) \geq \lambda_0 > 0$$

Regular Form:

$$\begin{bmatrix} \eta \\ \xi \end{bmatrix} = T(x), \quad \frac{\partial T}{\partial x} B(x) = \begin{bmatrix} 0 \\ I \end{bmatrix}$$

$$\dot{\eta} = f_a(\eta, \xi), \qquad \dot{\xi} = f_b(\eta, \xi) + G(x)u + \delta(t, x, u)$$

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Sliding Manifold:

$$s = \xi - \phi(\eta) = 0, \qquad \phi(0) = 0$$

$$s(t) \equiv 0 \quad \Rightarrow \quad \dot{\eta} = f_a(\eta, \phi(\eta))$$

Design ϕ s.t. the origin of $\dot{\eta} = f_a(\eta, \phi(\eta))$ is asymp. stable

$$\dot{s} = f_b(\eta, \xi) - \frac{\partial \phi}{\partial \eta} f_a(\eta, \xi) + G(x)u + \delta(t, x, u)$$

$$u = \psi(\eta, \xi) + v$$

Typical choices of ψ:

$$\psi = 0, \qquad \psi = -\hat{G}^{-1}[f_b - (\partial \phi / \partial \eta) f_a]$$

$$\dot{s} = G(x)v + \Delta(t, x, v)$$

$$\left\| \frac{\Delta(t, x, v)}{\lambda_{min}(G(x))} \right\| \leq \varrho(x) + \kappa_0 \|v\|, \quad \forall\, (t, x, v) \in [0, \infty) \times D \times R^m$$

$$\varrho(x) \geq 0, \quad 0 \leq \kappa_0 < 1 \quad \color{red}{(Known)}$$

$$V = \frac{1}{2}s^T s \;\Rightarrow\; \dot{V} = s^T \dot{s} = s^T G(x) v + s^T \Delta(t, x, v)$$

$$v = -\beta(x)\,\frac{s}{\|s\|}, \quad \beta(x) \geq \frac{\varrho(x)}{1-\kappa_0} + \beta_0,\; \beta_0 > 0$$

$$\begin{aligned}
\dot{V} &= -\beta(x) s^T G(x) s/\|s\| + s^T \Delta(t, x, v) \\
&\leq \lambda_{min}(G(x))[-\beta(x) + \varrho(x) + \kappa_0 \beta(x)]\;\|s\| \\
&= \lambda_{min}(G(x))[-(1-\kappa_0)\beta(x) + \varrho(x)]\;\|s\| \\
&\leq -\lambda_{min}(G(x))\beta_0(1-\kappa_0)\|s\| \\
&\leq -\lambda_0\beta_0(1-\kappa_0)\|s\| \;=\; -\lambda_0\beta_0(1-\kappa_0)\sqrt{2V}
\end{aligned}$$

Trajectories reach the manifold $s = 0$ in finite time and cannot leave it

Continuous Implementation

$$\text{Sat}(y) = \begin{cases} y, & \text{if } \|y\| \le 1 \\ y/\|y\|, & \text{if } \|y\| > 1 \end{cases}$$

$$v = -\beta(x)\ \text{Sat}\left(\frac{s}{\mu}\right)$$

$$\|s\| \ge \mu \;\Rightarrow\; \text{Sat}(s/\mu) = s/\|s\| \;\Rightarrow\; s^T\dot{s} \le -\lambda_0\beta_0(1-\kappa_0)\|s\|$$

Trajectories reach the boundary layer $\{\|s\| \le \mu\}$ in finite time and remains inside thereafter

Study the behavior of η: $\quad \dot{\eta} = f_a(\eta, \phi(\eta) + s)$

$$\alpha_1(\|\eta\|) \le V_0(\eta) \le \alpha_2(\|\eta\|)$$

$$\frac{\partial V_0}{\partial \eta} f_a(\eta, \phi(\eta) + s) \le -\alpha_3(\|\eta\|), \qquad \forall\ \|\eta\| \ge \alpha_4(\|s\|)$$

$$\|s\| \le c \Rightarrow \dot{V}_0 \le -\alpha_3(\|\eta\|), \quad \text{for } \|\eta\| \ge \alpha_4(c)$$

$$\alpha(r) = \alpha_2(\alpha_4(r))$$

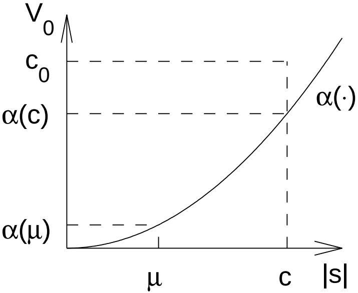
$$\begin{aligned} V_0(\eta) \ge \alpha(c) \quad &\Leftrightarrow \quad V_0(\eta) \ge \alpha_2(\alpha_4(c)) \ \Rightarrow \alpha_2(\|\eta\|) \ge \alpha_2(\alpha_4(c)) \\ &\Rightarrow \quad \|\eta\| \ge \alpha_4(c) \\ &\Rightarrow \quad \dot{V}_0 \le -\alpha_3(\|\eta\|) \le -\alpha_3(\alpha_4(c)) \end{aligned}$$

$$\Omega = \{V_0(\eta) \le c_0\} \times \{\|s\| \le c\}, \quad c_0 \ge \alpha(c), \ \Omega \subset T(D)$$

$$V_0(\eta) \geq \alpha(\mu) \;\Rightarrow\; \dot{V}_0 \leq -\alpha_3(\alpha_4(\mu))$$

$\Rightarrow \;\Omega_\mu = \{V_0(\eta) \leq \alpha(\mu)\} \times \{\|s\| \leq \mu\}$ is positively invariant

In summary, all trajectories starting in Ω remain in Ω and reach Ω_μ in finite time and remain inside thereafter



Theorem 10.1

Suppose all the assumptions hold over Ω. Then, for all $(\eta(0), \xi(0)) \in \Omega$, the trajectory $(\eta(t), \xi(t))$ is bounded for all $t \geq 0$ and reaches the positively invariant set Ω_μ in finite time. If the assumptions hold globally and $V(\eta)$ is radially unbounded, the foregoing conclusion holds for any initial state

Example 10.2 (Magnetic levitation - friction neglected)

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = 1 + \frac{m_o}{m}u, \quad x_1 \geq 0,\ -2 \leq u \leq 0$$

We want to stabilize the system at $x_1 = 1$. Nominal steady-state control is $u_{ss} = -1$

Shift the equilibrium point to the origin: $x_1 \to x_1 - 1,\ u \to u + 1$

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = \frac{m - m_o}{m} + \frac{m_o}{m}u$$

$$x_1 \geq -1, \quad |u| \leq 1$$

Assume $\left|\frac{(m - m_o)}{m_o}\right| \leq \frac{1}{3}$

$$s = x_1 + x_2 \quad \Rightarrow \quad \dot{x}_1 = -x_1 + s$$

$$V_0 = \frac{1}{2}x_1^2$$

$$\dot{V}_0 = -x_1^2 + x_1 s \leq -(1-\theta)x_1^2, \qquad \forall\ |x_1| \geq |s|/\theta, \quad 0 < \theta < 1$$

$$\alpha_1(r) = \alpha_2(r) = \frac{1}{2}r^2,\ \alpha_3(r) = (1-\theta)r^2,\ \alpha_4(r) = r/\theta$$

$$\alpha(r) = \alpha_2(\alpha_4(r)) = \frac{1}{2}(r/\theta)^2$$

With $c_0 = \alpha(c), \quad \Omega = \{|x_1| \leq c/\theta\} \times \{|s| \leq c\}$

$$\Omega_\mu = \{|x_1| \leq \mu/\theta\} \times \{|s| \leq \mu\}$$

$$\Omega = \{|x_1| \le c/\theta\} \times \{|s| \le c\}$$

Take $c \le \theta$ to meet the constraint $x_1 \ge -1$

$$\dot{s} = x_2 + \frac{m - m_o}{m} + \frac{m_o}{m} u$$

$$\left| \frac{x_2 + (m - m_o)/m}{m_o/m} \right| = \left| \frac{m}{m_o} x_2 + \frac{m - m_o}{m_o} \right| \le \tfrac{1}{3}(4|x_2| + 1)$$

In Ω, $\quad |x_2| \le |x_1| + |s| \le c(1 + 1/\theta)$

with $\dfrac{1}{\theta} = 1.1, \quad \left| \dfrac{x_2 + (m - m_o)/m}{m_o/m} \right| \le \dfrac{8.4c + 1}{3}$

To meet the constraint $|u| \leq 1$ limit c to

$$\frac{8.4c + 1}{3} < 1 \Leftrightarrow \ c < 0.238 \text{ and take } u = -\text{sat}\left(\frac{s}{\mu}\right)$$

With $c = 0.23$, Theorem 10.1 ensures that all trajectories starting in Ω stay in Ω and enter Ω_μ in finite time

$$\text{Inside } \Omega_\mu, \quad |x_1| \leq \mu/\theta = 1.1\mu$$

μ can be chosen small enough to meet any specified ultimate bound on x_1

$$\text{For } |x_1| \leq 0.01, \text{ take } \mu = 0.01/1.1 \approx 0.009$$

With further analysis inside Ω_μ we can derive a less conservative estimate of the ultimate bound of $|x_1|$. In Ω_μ, the closed-loop system is represented by

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = \frac{m - m_o}{m} - \frac{m_o(x_1 + x_2)}{m\mu}$$

which has a unique equilibrium point at

$$\left(x_1 = \frac{\mu(m - m_o)}{m_o},\ x_2 = 0 \right)$$

and its matrix is Hurwitz

$$\lim_{t\to\infty} x_1(t) = \frac{\mu(m - m_o)}{m_o}, \quad \lim_{t\to\infty} x_2(t) = 0$$

$$\left|\frac{(m - m_o)}{m_o}\right| \leq \frac{1}{3} \;\Rightarrow\; |x_1| \leq 0.34\mu$$

For $|x_1| \leq 0.01$, take $\mu = 0.029$

We can also obtain a less conservative estimate of the region of attraction

$$V_1 = \frac{1}{2}(x_1^2 + s^2)$$

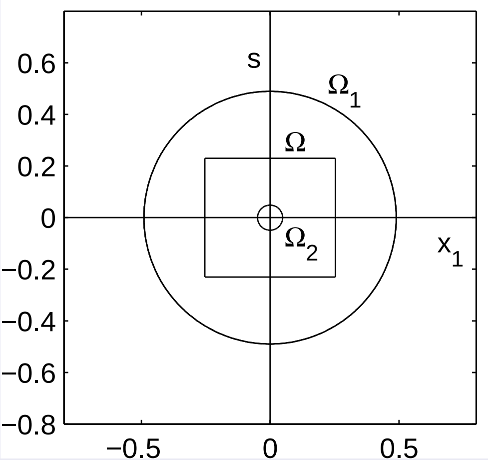
$$\dot{V}_1 \leq -x_1^2 + s^2 - \frac{m_o}{m}\left[1 - \left|\frac{m - m_o}{m_o}\right|\right]|s| \leq -x_1^2 + s^2 - \tfrac{1}{2}|s|$$

for $|s| \geq \mu$

$$\dot{V}_1 \leq -x_1^2 + s^2 + \left|\frac{m - m_o}{m_o}\right| |s| - \frac{m_o}{m}\frac{s^2}{\mu} \leq -x_1^2 + s^2 + \tfrac{1}{2}|s| - \frac{3s^2}{4\mu}$$

for $|s| \leq \mu$

With $\mu = 0.029$, it can be verified that $\dot{V}_1$ is less than a negative number in the set $\{0.0012 \leq V_1 \leq 0.12\}$. Therefore, all trajectories starting in $\Omega_1 = \{V_1 \leq 0.12\}$ enter $\Omega_2 = \{V_1 \leq 0.0012\}$ in finite time. Since $\Omega_2 \subset \Omega$, our earlier analysis holds and the ultimate bound of $|x_1|$ is 0.01. The new estimate of the region of attraction, Ω_1, is larger than Ω


0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−0.5
0
0.5
s
x1
Ω1
Ω
Ω2

Theorem 10.2

Suppose all the assumptions of Theorem 10.1 hold over Ω with

- $\varrho(0) = 0$
- The origin of $\dot{\eta} = f_a(\eta, \phi(\eta))$ is exponentially stale

Then there exits $\mu^* > 0$ such that for all $0 < \mu < \mu^*$, the origin of the closed-loop system is exponentially stable and Ω is a subset of its region of attraction. If the assumptions hold globally, the origin will be globally uniformly asymptotically stable

Proof

By Theorem 10.1, all trajectories starting in Ω enter Ω_μ in finite time. Inside Ω_μ

$$\dot{\eta} = f_a(\eta, \phi(\eta) + s), \qquad \mu\dot{s} = \beta(x)G(x)s + \mu\Delta(t, x, v)$$

By the converse Lyapunov theorem, there is $V_1(\eta)$ that satisfies

$$c_1\|\eta\|^2 \le V_1(\eta) \le c_2\|\eta\|^2$$

$$\frac{\partial V_1}{\partial \eta} f_a(\eta, \phi(\eta)) \le -c_3\|\eta\|^2$$

$$\left\|\frac{\partial V_1}{\partial \eta}\right\| \le c_4\|\eta\|$$

in some neighborhood N_η of $\eta = 0$

By the smoothness of f_a we have

$$\|f_a(\eta, \phi(\eta) + s) - f_a(\eta, \phi(\eta))\| \leq k_1 \|s\|$$

in some neighborhood N of $(\eta, \xi) = (0, 0)$

Choose μ small enough that $\Omega_\mu \subset N_\eta \cap N$. Inside Ω_μ

$$\begin{aligned}
s^T \dot{s} &= -\frac{\beta(x)}{\mu} s^T G(x) s + s^T \Delta(t, x, v) \\
&\leq -\frac{\beta \lambda_{min}(G)}{\mu} \|s\|^2 + \lambda_{min}(G) \left[\varrho + \frac{\kappa_0 \beta \|s\|}{\mu} \right] \|s\| \\
&\leq -\frac{\lambda_0 \beta_0 (1 - \kappa_0)}{\mu} \|s\|^2 + \lambda_{min}(G) \varrho \|s\|
\end{aligned}$$

Since G is continuous and ϱ is locally Lipschitz with $\varrho(0) = 0$, we arrive at

$$s^T \dot{s} \leq - \frac{\lambda_0 \beta_0 (1 - \kappa_0)}{\mu} \|s\|^2 + k_2 \|\eta\| \|s\| + k_3 \|s\|^2$$

$$W = V_1(\eta) + \tfrac{1}{2} s^T s$$

$$\begin{aligned} \dot{W} \;\; \leq \;\; & -c_3 \|\eta\|^2 + c_4 k_1 \|\eta\| \|s\| + k_2 \|\eta\| \|s\| \\ & + k_3 \|s\|^2 - \frac{\lambda_0 \beta_0 (1 - \kappa_0)}{\mu} \|s\|^2 \end{aligned}$$

$$\dot{W} \leq - \begin{bmatrix} \|\eta\| \\ \|s\| \end{bmatrix}^T \begin{bmatrix} c_3 & -\frac{c_4 k_1 + k_2}{2} \\ -\frac{c_4 k_1 + k_2}{2} & \frac{\lambda_0 \beta_0 (1-\kappa_0)}{\mu} - k_3 \end{bmatrix} \begin{bmatrix} \|\eta\| \\ \|s\| \end{bmatrix}$$

$$\mu < \frac{4 c_3 \lambda_0 \beta_0 (1 - \kappa_0)}{4 c_3 k_3 + (c_4 k_1 + k_2)^2}$$

The basic idea of the foregoing proof is that, inside the boundary layer, the control

$$v = - \frac{\beta(x) s}{\mu}$$

acts as high-gain feedback for small μ. By choosing μ small enough, the high-gain feedback stabilizes the origin

Unmatched Uncertainty

$$\dot{x} = f(x) + B(x)[G(x)u + \delta(t, x, u)] + \delta_1(x)$$

$$\begin{aligned} \dot{\eta} &= f_a(\eta, \xi) + \delta_a(\eta, \xi) \\ \dot{\xi} &= f_b(\eta, \xi) + G(x)u + \delta(t, x, u) + \delta_b(\eta, \xi) \end{aligned}$$

$$s = \xi - \phi(\eta)$$

Reduced-order model on the sliding manifold:

$$\dot{\eta} = f_a(\eta, \phi(\eta)) + \delta_a(\eta, \phi(\eta))$$

Design of ϕ to stabilize $\eta = 0$ in the presence of δ_a

Example 10.3

$$\dot{x}_1 = x_2 + \theta_1 x_1 \sin x_2, \qquad \dot{x}_2 = \theta_2 x_2^2 + x_1 + u$$

$$|\theta_1| \le a, \quad |\theta_2| \le b$$

$$x_2 = -kx_1 \quad \Rightarrow \quad \dot{x}_1 = -kx_1 + \theta_1 x_1 \sin x_2$$

$$V_1 = \tfrac{1}{2}x_1^2 \quad \Rightarrow \quad x_1\dot{x}_1 \le -kx_1^2 + ax_1^2$$

$$s = x_2 + kx_1, \qquad k > a$$

$$\dot{s} = \theta_2 x_2^2 + x_1 + u + k(x_2 + \theta_1 x_1 \sin x_2)$$

$$u = -x_1 - kx_2 + v \quad \Rightarrow \quad \dot{s} = v + \Delta(x)$$

$$\Delta(x) = \theta_2 x_2^2 + k\theta_1 x_1 \sin x_2$$

$$\Delta(x) = \theta_2 x_2^2 + k\theta_1 x_1 \sin x_2$$

$$|\Delta(x)| \leq ak|x_1| + bx_2^2$$

$$\beta(x) = ak|x_1| + bx_2^2 + \beta_0, \qquad \beta_0 > 0$$

By Theorem 10.2,

$$u = -x_1 - kx_2 - \beta(x) \ \mathrm{sat}\left(\frac{s}{\mu}\right)$$

with sufficiently small μ, globally stabilizes the origin

Example 10.4

$$\dot{x}_1 = x_1 + (1 - \theta_1)x_2, \qquad \dot{x}_2 = \theta_2 x_2^2 + x_1 + u$$

$$|\theta_1| \le a, \quad |\theta_2| \le b$$

$$\dot{x}_1 = x_1 + (1 - \theta_1)x_2$$

Design x_2 to robustly stabilize the origin $x_1 = 0$

We must have $|a| < 1$

$$x_2 = -kx_1 \;\Rightarrow\; x_1\dot{x}_1 = x_1^2 - k(1 - \theta_1)x_1^2 \le -[k(1-a) - 1]x_1^2$$

$$k > \frac{1}{1-a}$$

$$s = x_2 + kx_1$$

Proceeding as in the previous example, we end up with

$$u = -(1+k)x_1 - kx_2 - \beta(x)\,\mathrm{sat}(s/\mu)$$

$$\beta(x) = bx_2^2 + ak|x_2| + \beta_0, \quad \beta_0 > 0$$

Alternative Approach: Suppose $G(x)$ is a diagonal matrix with positive elements

$$\dot{s} = G(x)v + \Delta(t, x, v)$$

$$\dot{s}_i = g_i(x)v_i + \Delta_i(t, x, v), \quad 1 \le i \le m$$

$$g_i(x) \ge g_0 > 0 \quad \left| \frac{\Delta_i(t, x, v)}{g_i(x)} \right| \le \varrho(x) + \kappa_0 \max_{1 \le i \le m} |v_i|$$

$$v_i = -\beta(x) \,\mathrm{sat}(s_i/\mu), \quad 1 \le i \le m$$

$$V_i = \frac{1}{2} s_i^2$$

$$
\begin{aligned}
\dot{V}_i &= s_i g_i(x) v_i + s_i \Delta_i(t,x,v) \\
&\le g_i(x)\{s_i v_i + |s_i|[\varrho(x) + \kappa_0 \max_{1\le i\le m} |v_i|]\} \\
&\le g_i(x)[-\beta(x) + \varrho(x) + \kappa_0 \beta(x)]|s_i| \\
&= g_i(x)[-(1-\kappa_0)\beta(x) + \varrho(x)]|s_i| \\
&\le g_i(x)[-\varrho(x) - (1-\kappa_0)\beta_0 + \varrho(x)]|s_i| \\
&\le -g_0 \beta_0 (1-\kappa_0)|s_i|
\end{aligned}
$$

$$
\dot{V}_i \le -g_0 \beta_0 (1-\kappa_0)\sqrt{2V_i}
$$

ensures that all trajectories reach the boundary layer

$$
\{|s_i| \le \mu,\ 1 \le i \le m\}
$$

in finite time

Results similar to Theorems 10.1 and 10.2 can be proved with

$$\Omega = \{V_0(\eta) \le c_0\} \times \{|s_i| \le c,\ 1 \le i \le m\}$$

$$c_0 \ge \alpha(c), \quad \alpha(r) = \alpha_2(\alpha_4(r\sqrt{m}))$$

$$\Omega_\mu = \{V_0(\eta) \le \alpha(\mu)\} \times \{|s_i| \le \mu,\ 1 \le i \le m\}$$