Nonlinear Control

Lecture # 12

Nonlinear Observers

and

Output Feedback Stabilization

Local Observers

$$\dot{x} = f(x, u), \qquad y = h(x)$$

$$\dot{\hat{x}} = f(\hat{x}, u) + H[y - h(\hat{x})]$$

$$\tilde{x} = x - \hat{x}$$

$$\dot{\tilde{x}} = f(x, u) - f(\hat{x}, u) - H[h(x) - h(\hat{x})]$$

We seek a local solution for sufficiently small $\|\tilde{x}(0)\|$

Linearization at $\tilde{x} = 0$:

$$\dot{\tilde{x}} = \left[\frac{\partial f}{\partial x}(x(t), u(t)) - H\frac{\partial h}{\partial x}(x(t))\right]\tilde{x}$$

Steady-state solution:

$$0 = f(x_{ss}, u_{ss}), \qquad 0 = h(x_{ss})$$

Assumption: given $\varepsilon > 0$, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$\|x(0) - x_{ss}\| \le \delta_1 \quad \text{and} \quad \|u(t) - u_{ss}\| \le \delta_2 \ \forall\ t \ge\ 0$$

$$\Rightarrow \qquad \|x(t) - x_{ss}\| \le \varepsilon \ \forall\ t \ge\ 0$$

$$A = \frac{\partial f}{\partial x}(x_{ss}, u_{ss}), \qquad C = \frac{\partial h}{\partial x}(x_{ss})$$

Assume that (A, C) is detectable. Design H such that $A - HC$ is Hurwitz

Lemma 11.1

For sufficiently small $\|\tilde{x}(0)\|$, $\|x(0) - x_{ss}\|$, and $\sup_{t\geq 0} \|u(t) - u_{ss}\|$,

$$\lim_{t\to\infty} \tilde{x}(t) = 0$$

Proof

$$f(x,u) - f(\hat{x},u) = \int_0^1 \frac{\partial f}{\partial x}(x - \sigma\tilde{x}, u)\ d\sigma\ \tilde{x}$$

$$\|f(x,u) - f(\hat{x},u) - A\tilde{x}\| =$$

$$\left\|\int_0^1 \left[\frac{\partial f}{\partial x}(x - \sigma\tilde{x}, u) - \frac{\partial f}{\partial x}(x,u) + \frac{\partial f}{\partial x}(x,u) - \frac{\partial f}{\partial x}(x_{ss}, u_{ss})\right] d\sigma\ \tilde{x}\right\|$$

$$\leq\ L_1\left(\tfrac{1}{2}\|\tilde{x}\| + \|x - x_{ss}\| + \|u - u_{ss}\|\right)\|\tilde{x}\|$$

$$\|h(x) - h(\hat{x}) - C\tilde{x}\| \leq L_2(\tfrac{1}{2}\|\tilde{x}\| + \|x - x_{ss}\|)\|\tilde{x}\|$$

$$\dot{\tilde{x}} = (A - HC)\tilde{x} + \Delta(x, u, \tilde{x})$$

$$\|\Delta(x, u, \tilde{x})\| \leq k_1\|\tilde{x}\|^2 + k_2(\varepsilon + \delta_2)\|\tilde{x}\|$$

$$P(A - HC) + (A - HC)^T P = -I$$

$$V = \tilde{x}^T P \tilde{x}$$

$$\dot{V} \leq -\|\tilde{x}\|^2 + c_4 k_1\|\tilde{x}\|^3 + c_4 k_2(\varepsilon + \delta_2)\|\tilde{x}\|^2$$

$$\dot{V} \leq -\tfrac{1}{3}\|\tilde{x}\|^2, \qquad \text{for } c_4 k_1\|\tilde{x}\| \leq \tfrac{1}{3} \quad \text{and} \quad c_4 k_2(\varepsilon + \delta_2) \leq \tfrac{1}{3}$$

For sufficiently small $\|\tilde{x}(0)\|$, ε, and δ_2, the estimation error converges to zero as t tends to infinity

The Extended Kalman Filter

$$\dot{x} = f(x, u), \qquad y = h(x)$$

$$\dot{\hat{x}} = f(\hat{x}, u) + H(t)[y - h(\hat{x})]$$

$$\tilde{x} = x - \hat{x}$$

$$\dot{\tilde{x}} = f(x, u) - f(\hat{x}, u) - H(t)[h(x) - h(\hat{x})]$$

Expand the right-hand side in a Taylor series about $\tilde{x} = 0$ and evaluate the Jacobian matrices along $\hat{x}$

$$\dot{\tilde{x}} = [A(t) - H(t)C(t)]\tilde{x} + \Delta(\tilde{x}, x, u)$$

$$A(t) = \frac{\partial f}{\partial x}(\hat{x}(t), u(t)), \qquad C(t) = \frac{\partial h}{\partial x}(\hat{x}(t))$$

Kalman Filter Gain:

$$H(t) = P(t)C^T(t)R^{-1}$$

$$\dot{P} = AP + PA^T + Q - PC^TR^{-1}CP, \qquad P(t_0) = P_0$$

P_0, Q and R are symmetric, positive definite matrices

Assumption 11.1: $P(t)$ exists for all $t \geq t_0$ and satisfies

$$\alpha_1 I \leq P(t) \leq \alpha_2 I, \qquad \alpha_i > 0$$

Remarks:

- Assumption 11.1 cannot be checked offline
- The observer and Riccati equations are solved simultaneously in real time

Lemma 11.2

There exist positive constants c, k, and λ such that

$$\|\tilde{x}(0)\| \le c \quad \Rightarrow \quad \|\tilde{x}(t)\| \le k e^{-\lambda(t-t_0)}, \quad \forall\, t \ge t_0 \ge 0$$

Proof

$$\|f(x,u) - f(\hat{x},u) - A(t)\tilde{x}\| =$$

$$\left\| \int_0^1 \left[\frac{\partial f}{\partial x}(\sigma\tilde{x} + \hat{x}, u) - \frac{\partial f}{\partial x}(\hat{x}, u) \right] d\sigma\ \tilde{x} \right\| \le \ \tfrac{1}{2} L_1 \|\tilde{x}\|^2$$

$$\|h(x) - h(\hat{x}) - C(t)\tilde{x}\| =$$

$$\left\| \int_0^1 \left[\frac{\partial h}{\partial x}(\sigma\tilde{x} + \hat{x}) - \frac{\partial h}{\partial x}(\hat{x}) \right] d\sigma\ \tilde{x} \right\| \le \ \tfrac{1}{2} L_2 \|\tilde{x}\|^2$$

$$\|C(t)\| = \left\| \frac{\partial h}{\partial x}(x - \tilde{x}) \right\| \leq \left\| \frac{\partial h}{\partial x}(0) \right\| + L_2(\|x\| + \|\tilde{x}\|)$$

$$\|\Delta(\tilde{x}, x, u)\| \leq k_1 \|\tilde{x}\|^2 + k_3 \|\tilde{x}\|^3$$

$$\alpha_1 I \leq P(t) \leq \alpha_2 I \;\Leftrightarrow\; \alpha_3 I \leq P^{-1}(t) \leq \alpha_4 I, \; \alpha_i > 0$$

$$V = \tilde{x}^T P^{-1} \tilde{x}$$

$$\frac{d}{dt} P^{-1} = -P^{-1} \dot{P} P^{-1}$$

$$\begin{aligned}
\dot{V} &= \tilde{x}^T P^{-1}\dot{\tilde{x}} + \dot{\tilde{x}}^T P^{-1}\tilde{x} + \tilde{x}^T \frac{d}{dt}P^{-1}\tilde{x} \\
&= \tilde{x}^T P^{-1}(A - PC^T R^{-1}C)\tilde{x} \\
&\quad + \tilde{x}^T(A^T - C^T R^{-1}CP)P^{-1}\tilde{x} \\
&\quad - \tilde{x}^T P^{-1}\dot{P}P^{-1}\tilde{x} + 2\tilde{x}^T P^{-1}\Delta \\
&= \tilde{x}^T P^{-1}(AP + PA^T - PC^T R^{-1}CP - \dot{P})P^{-1}\tilde{x} \\
&\quad - \tilde{x}^T C^T R^{-1}C\tilde{x} + 2\tilde{x}^T P^{-1}\Delta \\
&= -\tilde{x}^T(P^{-1}QP^{-1} + C^T R^{-1}C)\tilde{x} + 2\tilde{x}^T P^{-1}\Delta
\end{aligned}$$

$$\dot{V} \le -c_1\|\tilde{x}\|^2 + c_2 k_1 \|\tilde{x}\|^3 + c_2 k_2 \|\tilde{x}\|^4$$

$$\dot{V} \le -\tfrac{1}{2}c_1\|\tilde{x}\|^2, \qquad \text{for} \quad \|\tilde{x}\| \le r$$

Example 11.1

$$\dot{x} = A_1 x + B_1[0.25 x_1^2 x_2 + 0.2 \sin 2t], \quad y = C_1 x$$

$$A_1 = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Investigate boundedness of $x(t)$

$$P_1 A_1 + A_1^T P_1 = -I \quad \Rightarrow \quad P_1 = \frac{1}{2}\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$$

$$V(x) = x^T P_1 x$$

$$
\begin{aligned}
\dot{V} &= -x^T x + 2x^T P_1 B_1 [0.25 x_1^2 x_2 + 0.2 \sin 2t] \\
&\le -\|x\|^2 + 0.5\|P_1 B_1\| x_1^2 \|x\|^2 + 0.4\|P_1 B_1\| \|x\| \\
&= -\|x\|^2 + \frac{x_1^2}{2\sqrt{2}}\|x\|^2 + \frac{0.4}{\sqrt{2}}\|x\| \\
&\le -0.5\|x\|^2 + \frac{0.4}{\sqrt{2}}\|x\|, \quad \text{for } \; x_1^2 \le \sqrt{2}
\end{aligned}
$$

$$
\frac{\sqrt{2}}{\begin{bmatrix} 1 & 0 \end{bmatrix} P_1^{-1} \begin{bmatrix} 1 & 0 \end{bmatrix}^T} = \sqrt{2}
$$

$$
\Omega = \{V(x) \le \sqrt{2}\} \subset \{x_1^2 \le \sqrt{2}\}
$$

Inside Ω

$$\begin{aligned}\dot{V} &\le -0.5\|x\|^2 + \frac{0.4}{\sqrt{2}}\|x\| \\ &\le -0.15\|x\|^2, \quad \forall\ \|x\| \ge \frac{0.4}{0.35\sqrt{2}} = 0.8081\end{aligned}$$

$$\lambda_{max}(P_1) = 1.7071 \quad \Rightarrow \quad (0.8081)^2\lambda_{max}(P_1) < \sqrt{2}$$

$$\Rightarrow \quad \{\|x\| \le 0.8081\} \subset \Omega$$

$\Rightarrow \quad \Omega$ is positively invariant

Design EKF to estimate $x(t)$ for $x(0) \in \Omega$

$$A(t) = \begin{bmatrix} 0 & 1 \\ -1 + 0.5\hat{x}_1(t)\hat{x}_2(t) & -2 + 0.25\hat{x}_1^2(t) \end{bmatrix}$$

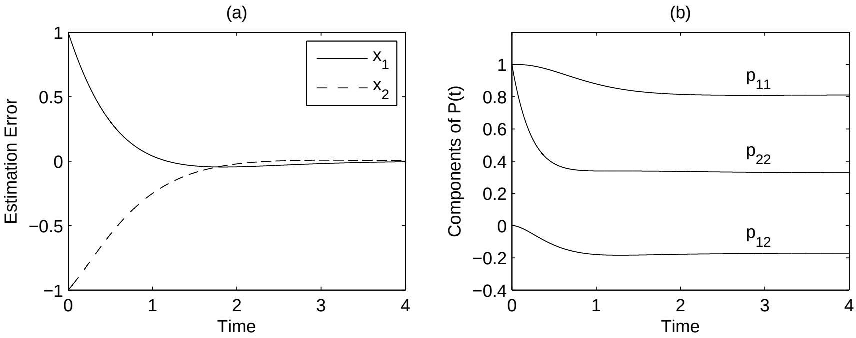
$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$Q = R = P(0) = I$$

$$\dot{P} = AP + PA^T + I - PC^TCP, \qquad P(0) = I$$

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$$

$$
\begin{aligned}
\dot{\hat{x}}_1 &= \hat{x}_2 + p_{11}(y - \hat{x}_1) \\
\dot{\hat{x}}_2 &= -\hat{x}_1 - 2\hat{x}_2 + 0.25\hat{x}_1^2\hat{x}_2 + 0.2\sin 2t + p_{12}(y - \hat{x}_1) \\
\dot{p}_{11} &= 2p_{12} + 1 - p_{11}^2 \\
\dot{p}_{12} &= p_{11}(-1 + 0.5\hat{x}_1\hat{x}_2) + p_{12}(-2 + 0.25\hat{x}_1^2) \\
&\quad + p_{22} - p_{11}p_{12} \\
\dot{p}_{22} &= 2p_{12}(-1 + 0.5\hat{x}_1\hat{x}_2) + 2p_{22}(-2 + 0.25\hat{x}_1^2) \\
&\quad + 1 - p_{12}^2
\end{aligned}
$$



(a)
1
0.5
0
−0.5
−1
0
1
2
3
4
Time
Estimation Error
x1
x2
(b)
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
0
1
2
3
4
Time
Components of P(t)
p11
p22
p12

Global Observers

Observer Form:

$$\dot{x} = Ax + \psi(u, y), \qquad y = Cx$$

(A, C) observable

$$\dot{\hat{x}} = A\hat{x} + \psi(u, y) + H(y - C\hat{x})$$

$$\tilde{x} = x - \hat{x}$$

$$\dot{\tilde{x}} = (A - HC)\tilde{x}$$

Design H such that $A - HC$ is Hurwitz

$$lim_{t\to\infty}\tilde{x}(t) = 0, \qquad \forall\ \tilde{x}(0)$$

$$\dot{x} = Ax + \psi(u, y) + \phi(x, u), \qquad y = Cx$$

$$\|\phi(x, u) - \phi(z, u)\| \leq L\|x - z\|$$

$$\dot{\hat{x}} = A\hat{x} + \psi(u, y) + \phi(\hat{x}, u) + H(y - C\hat{x})$$

$$\dot{\tilde{x}} = (A - HC)\tilde{x} + \phi(x, u) - \phi(\hat{x}, u)$$

$$P(A - HC) + (A - HC)^T P = -I, \qquad V = \tilde{x}^T P \tilde{x}$$

$$\dot{V} = -\tilde{x}^T \tilde{x} + 2\tilde{x}^T P[\phi(x, u) - \phi(\hat{x}, u)] \leq -\|\tilde{x}\|^2 + 2L\|P\|\|\tilde{x}\|^2$$

$$L < \frac{1}{2\|P\|} \quad \Rightarrow \quad lim_{t\to\infty}\tilde{x}(t) = 0, \qquad \forall\ \tilde{x}(0)$$

High-Gain Observers

Example 11.2

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = \phi(x,u), \qquad y = x_1$$

$$\dot{\hat{x}}_1 = \hat{x}_2 + h_1(y - \hat{x}_1), \qquad \dot{\hat{x}}_2 = \phi_0(\hat{x},u) + h_2(y - \hat{x}_1)$$

$$|\phi_0(z,u) - \phi(x,u)| \le L\|x - z\| + M$$

$$\dot{\tilde{x}} = A_o\tilde{x} + B\delta(x,\tilde{x},u)$$

$$A_o = \begin{bmatrix} -h_1 & 1 \\ -h_2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \delta = \phi(x,u) - \phi_0(\hat{x},u)$$

Design h_1 and h_2 such that A_o is Hurwitz

Transfer function from δ to $\tilde{x}$:

$$G_o(s) = \frac{1}{s^2 + h_1 s + h_2} \left[\begin{array}{c} 1 \\ s + h_1 \end{array} \right]$$

We can make $\sup_{\omega \in R} \|G_o(j\omega)\|$ arbitrarily small by choosing $h_2 \gg h_1 \gg 1$

$$h_1 = \frac{\alpha_1}{\varepsilon}, \qquad h_2 = \frac{\alpha_2}{\varepsilon^2}, \qquad \varepsilon \ll 1$$

$$G_o(s) = \frac{\varepsilon}{(\varepsilon s)^2 + \alpha_1 \varepsilon s + \alpha_2} \left[\begin{array}{c} \varepsilon \\ \varepsilon s + \alpha_1 \end{array} \right]$$

$$\lim_{\varepsilon \to 0} G_o(s) = 0$$

$$\eta_1 = \frac{\tilde{x}_1}{\varepsilon}, \qquad \eta_2 = \tilde{x}_2$$

$$\varepsilon\dot{\eta} = F\eta + \varepsilon B\delta, \qquad \text{where} \quad F = \begin{bmatrix} -\alpha_1 & 1 \\ -\alpha_2 & 0 \end{bmatrix}$$

$$PF + F^T P = -I, \qquad V = \eta^T P\eta$$

$$|\delta| \le L\|\tilde{x}\| + M \le L\|\eta\| + M$$

$$\begin{aligned} \varepsilon\dot{V} &= -\eta^T\eta + 2\varepsilon\eta^T PB\delta \\ &\le -\|\eta\|^2 + 2\varepsilon L\|PB\|\|\eta\|^2 + 2\varepsilon M\|PB\|\|\eta\| \end{aligned}$$

$$\varepsilon L\|PB\| \le \tfrac{1}{4} \quad \Rightarrow \quad \varepsilon\dot{V} \le -\tfrac{1}{2}\|\eta\|^2 + 2\varepsilon M\|PB\|\|\eta\|$$

$$\|\eta(t)\| \le \max\left\{ke^{-at/\varepsilon}\|\eta(0)\|,\ \varepsilon c M\right\}, \qquad \forall\ t \ge 0$$

Peaking Phenomenon:

$$x_1(0) \ne \hat{x}_1(0) \quad \Rightarrow \quad \eta_1(0) = O(1/\varepsilon)$$

The solution will contain a term of the form $(1/\varepsilon)e^{-at/\varepsilon}$

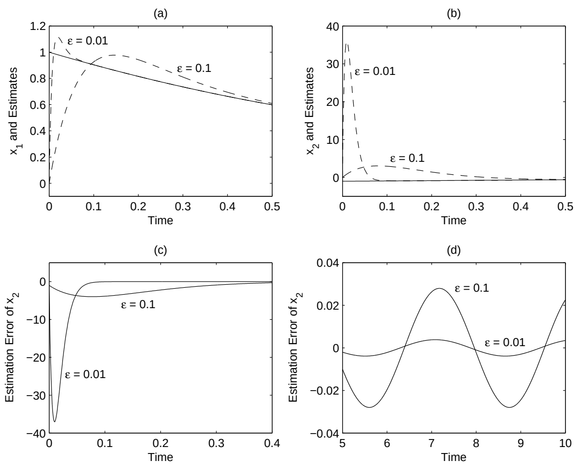
Example 11.1 (revisited)

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1 - 2x_2 + ax_1^2 x_2 + b \sin 2t, \qquad y = x_1$$

$a = 0.25$, $b = 0.2$, and $\Omega = \{1.5x_1^2 + x_1 x_2 + 0.5x_2^2 \leq \sqrt{2}\}$ is positively invariant

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + \frac{2}{\varepsilon}(y - \hat{x}_1) \\ \dot{\hat{x}}_2 &= -\hat{x}_1 - 2\hat{x}_2 + \hat{a}\hat{x}_1^2 \hat{x}_2 + \hat{b} \sin 2t + \frac{1}{\varepsilon^2}(y - \hat{x}_1) \end{aligned}$$

- Case 1: $\hat{a} = 0.25$ and $\hat{b} = 0.2$ (Figures (a) and (b))
- Case 2: $\hat{a} = \hat{b} = 0$ (Figures (c) and (d))


(a)
x_1 and Estimates
ε = 0.01
ε = 0.1
Time
(b)
x_2 and Estimates
ε = 0.01
ε = 0.1
Time
(c)
Estimation Error of x_2
ε = 0.1
ε = 0.01
Time
(d)
Estimation Error of x_2
ε = 0.1
ε = 0.01
Time

General case:

$$\begin{aligned} \dot{w} &= f_0(w, x, u) \\ \dot{x}_i &= x_{i+1} + \psi_i(x_1, \ldots, x_i, u), \qquad \text{for } 1 \le i \le \rho - 1 \\ \dot{x}_\rho &= \phi(w, x, u) \\ y &= x_1 \end{aligned}$$

$$|\psi_i(x_1, \ldots, x_i, u) - \psi_i(z_1, \ldots, z_i, u)| \le L_i \sum_{k=1}^{i} |x_k - z_k|$$

$$\begin{aligned} \dot{\hat{x}}_i &= \hat{x}_{i+1} + \psi_1(\hat{x}_1, \ldots, \hat{x}_i, u) + \frac{\alpha_i}{\varepsilon^i}(y - \hat{x}_1) \\ \dot{\hat{x}}_\rho &= \phi_0(\hat{x}, u) + \frac{\alpha_\rho}{\varepsilon^\rho}(y - \hat{x}_1) \end{aligned}$$

α_1 to α_ρ are chosen such that the roots of

$$s^\rho + \alpha_1 s^{\rho-1} + \cdots + \alpha_{\rho-1} s + \alpha_\rho = 0$$

have negative real parts

$$|\phi(w,x,u) - \phi_0(x,u)| \leq M$$

$$|\phi_0(x,u) - \phi_0(\hat{x},u)| \leq L\|x - \hat{x}\|$$

$$|\phi(w,x,u) - \phi_0(\hat{x},u)| \leq L\|x - \hat{x}\| + M$$

Lemma 11.3

For sufficiently small ε

$$|\tilde{x}_i| \leq \max\left\{\frac{b}{\varepsilon^{i-1}}\ e^{-at/\varepsilon},\ \varepsilon^{\rho+1-i} cM\right\}$$

Proof

$$\eta_1 = \frac{x_1 - \hat{x}_1}{\varepsilon^{\rho-1}}, \quad \ldots \quad , \quad \eta_{\rho-1} = \frac{x_{\rho-1} - \hat{x}_{\rho-1}}{\varepsilon}, \quad \eta_\rho = x_\rho - \hat{x}_\rho$$

$$\varepsilon\dot{\eta} = F\eta + \varepsilon\delta(w, x, \tilde{x}, u)$$

$$\delta = \mathrm{col}(\delta_1, \delta_2, \cdots, \delta_\rho), \quad \delta_\rho = \phi(w, x, u) - \phi_0(\hat{x}, u)$$

$$|\delta_i| \le L_i \sum_{k=1}^{i} \varepsilon^{i-k}|\eta_k|, \quad 1 \le i \le \rho - 1$$

$$F = \begin{bmatrix} -\alpha_1 & 1 & 0 & \cdots & 0 \\ -\alpha_2 & 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ -\alpha_{\rho-1} & & & 0 & 1 \\ -\alpha_\rho & 0 & \cdots & \cdots & 0 \end{bmatrix}$$

For $\varepsilon \le \varepsilon^*$

$$\|\delta\| \le L_\delta \|\eta\| + M$$

$$V = \eta^T P \eta, \quad PF + F^T P = -I$$

$$\varepsilon \dot{V} = -\eta^T \eta + 2\varepsilon \eta^T P \delta$$

$$\varepsilon \dot{V} \le -\|\eta\|^2 + 2\varepsilon \|P\| L_\delta \|\eta\|^2 + 2\varepsilon \|P\| M \|\eta\|$$

For $\epsilon \|P\| L_\delta \le \frac{1}{4}$,

$$\varepsilon \dot{V} \le -\tfrac{1}{2}\|\eta\|^2 + 2\varepsilon \|P\| M \|\eta\| \le -\tfrac{1}{4}\|\eta\|^2, \ \forall \ \|\eta\| \ge 8\varepsilon \|P\| M$$

By Theorem 4.5, $\quad \|\eta(t)\| \le \max\left\{ k e^{-at/\varepsilon} \|\eta(0)\|, \ \varepsilon c M \right\}$

$$|\tilde{x}_i| \le \varepsilon^{\rho - i} |\eta_i|$$

Output Feedback Stabilization: Linearization

$$\dot{x} = f(x,u), \quad y = h(x), \quad f(0,0) = 0, \quad h(0) = 0$$

$$\dot{x} = Ax + Bu \qquad y = Cx$$

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0,u=0}, \quad B = \left.\frac{\partial f}{\partial u}\right|_{x=0,u=0}, \quad C = \left.\frac{\partial h}{\partial x}\right|_{x=0}$$

Design a linear output feedback controller:

$$\dot{z} = Fz + Gy, \qquad u = Lz + My$$

such that the closed-loop matrix

$$\begin{bmatrix} A + BMC & BL \\ GC & F \end{bmatrix} \qquad \text{is Hurwitz}$$

Closed-loop system:

$$\dot{x} = f(x, Lz + Mh(x)), \qquad \dot{z} = Fz + Gh(x)$$

Linearization at the origin $(x = 0,\ z = 0)$ results in the Hurwitz matrix

$$\begin{bmatrix} A + BMC & BL \\ GC & F \end{bmatrix}$$

By Theorem 3.2, the origin is exponentially stable