Nonlinear Control

Lecture # 13

Output Feedback Stabilization

Passivity-Based Control

In Section 9.6 we saw that if the system

$$\dot{x} = f(x, u), \qquad y = h(x)$$

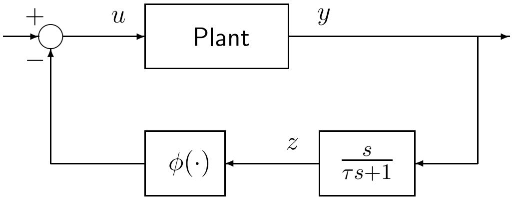
is passive (with a positive definite storage function) and zero-state observable, it can be stabilized by

$$u = -\phi(y), \quad \phi(0) = 0, \quad y^T\phi(y) > 0, \quad \forall\ y \neq 0$$

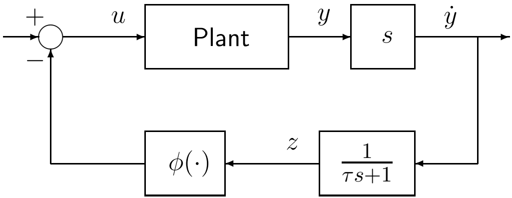
Suppose the system

$$\dot{x} = f(x, u), \qquad \dot{y} = \frac{\partial h}{\partial x} f(x, u) \stackrel{\text{def}}{=} \tilde{h}(x, u)$$

is passive (with a positive definite storage function $V(x)$) and zero state observable



+
−
u
Plant
y
z
$\phi(\cdot)$
$\frac{s}{\tau s+1}$



+
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$\phi(\cdot)$
$\frac{1}{\tau s+1}$

$$\frac{s}{\tau s + 1}$$

$$\tau \dot{w} = -w + y, \qquad z = (-w + y)/\tau$$

MIMO systems

$$\tau_i \dot{w}_i = -w_i + y_i, \quad z_i = (-w_i + y_i)/\tau_i, \quad \text{for } 1 \le i \le m$$

Note that

$$\tau_i \dot{z}_i = -z_i + \dot{y}_i$$

Lemma 12.1

Consider the system

$$\dot{x} = f(x, u), \qquad y = h(x)$$

and the output feedback controller

$$u_i = -\phi_i(z_i), \quad \tau_i \dot{w}_i = -w_i + y_i, \quad z_i = (-w_i + y_i)/\tau_i$$

$$\tau_i > 0,\ \phi_i(0) = 0,\ z_i\phi_i(z_i) > 0\ \forall\ z_i \neq 0$$

Suppose the auxiliary system

$$\dot{x} = f(x, u), \qquad \dot{y} = \tilde{h}(x, u)$$

is

- passive with a positive definite storage function $V(x)$

$$u^T \dot{y} \geq \dot{V} = \frac{\partial V}{\partial x} f(x, u), \qquad \forall \ (x, u)$$

- zero-state observable

$$\text{with } u = 0, \ \dot{y}(t) \equiv 0 \ \Rightarrow \ x(t) \equiv 0$$

Then the origin of the closed-loop system is asymptotically stable. It is globally asymptotically stable if $V(x)$ is radially unbounded and $\int_0^{z_i} \phi_i(\sigma) \ d\sigma \to \infty$ as $|z_i| \to \infty$

Proof

$$W(x,z) = V(x) + \sum_{i=1}^{m} \tau_i \int_0^{z_i} \phi_i(\sigma)\ d\sigma$$

$$\dot{W} = \dot{V} + \sum_{i=1}^{m} \tau_i \phi_i(z_i)\dot{z}_i \le u^T \dot{y} - \sum_{i=1}^{m} z_i \phi_i(z_i) - u^T \dot{y}$$

$$\dot{W} \le -\sum_{i=1}^{m} z_i \phi_i(z_i)$$

$$\dot{W} \equiv 0 \quad \Rightarrow \quad z(t) \equiv 0 \quad \Rightarrow \quad u(t) \equiv 0 \quad \text{and} \quad \dot{y}(t) \equiv 0$$

Apply the invariance principle

Example 12.2 (m-link Robot Manipulator)

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + D\dot{q} + g(q) = u$$

$$M = M^T > 0,\ (\dot{M} - 2C)^T = -(\dot{M} - 2C),\ D = D^T \geq 0$$

Stabilize the system at $q = q_r$, $e = q - q_r$, $\dot{e} = \dot{q}$

$$M(q)\ddot{e} + C(q,\dot{q})\dot{e} + D\dot{e} + g(q) = u$$

$$u = g(q) - K_p e + v, \quad [K_p = K_p > 0]$$

$$M(q)\ddot{e} + C(q,\dot{q})\dot{e} + D\dot{e} + K_p e = v, \qquad y = e$$

$$V = \tfrac{1}{2}\dot{e}^T M(q)\dot{e} + \tfrac{1}{2}e^T K_p e$$

$$V = \tfrac{1}{2}\dot{e}^T M(q)\dot{e} + \tfrac{1}{2}e^T K_p e$$

$$\dot{V} = \tfrac{1}{2}\dot{e}^T(\dot{M} - 2C)\dot{e} - \dot{e}^T D\dot{e} - \dot{e}^T K_p e + \dot{e}^T v + e^T K_p \dot{e} \;\le\; \dot{e}^T v$$

Is it zero-state observable? Set $v = 0$

$$\dot{e}(t) \equiv 0 \;\Rightarrow\; \ddot{e}(t) \equiv 0 \;\Rightarrow\; K_p e(t) \equiv 0 \;\Rightarrow\; e(t) \equiv 0$$

$$\tau_i \dot{w}_i = -w_i + e_i, \quad z_i = (-a_i w_i + e_i)/\tau_i, \quad \text{for } 1 \le i \le m$$

$$u = g(q) - K_p(q - q_r) - K_d z$$

K_d is positive diagonal matrix. Compare with state feedback

$$u = g(q) - K_p(q - q_r) - K_d \dot{q}$$

Observer-Based Control

$$\dot{x} = f(x, u), \qquad y = h(x)$$

State Feedback Controller: Design a locally Lipschitz $u = \gamma(x)$ to stabilize the origin of

$$\dot{x} = f(x, \gamma(x))$$

Observer:

$$\dot{\hat{x}} = f(\hat{x}, u) + H[y - h(\hat{x})]$$

$$\tilde{x} = x - \hat{x}$$

$$\dot{\tilde{x}} = f(x, u) - f(\hat{x}, u) - H[h(x) - h(\hat{x})] \stackrel{\text{def}}{=} g(x, \tilde{x})$$

$$g(x, 0) = 0$$

Design H such that $\dot{\tilde{x}} = g(x, \tilde{x})$ has an exponentially stable equilibrium point at $\tilde{x} = 0$ and there is Lyapunov function $V_1(\tilde{x})$ such that

$$c_1\|\tilde{x}\|^2 \leq V_1 \leq c_2\|\tilde{x}\|^2, \; \frac{\partial V_1}{\partial \tilde{x}} g \leq -c_3\|\tilde{x}\|^2, \; \left\|\frac{\partial V_1}{\partial \tilde{x}}\right\| \leq c_4\|\tilde{x}\|$$

$$u = \gamma(\hat{x})$$

Closed-loop system:

$$\dot{x} = f(x, \gamma(x - \tilde{x})), \qquad \dot{\tilde{x}} = g(x, \tilde{x}) \qquad (\star)$$

Theorem 12.1

- If the origin of $\dot{x} = f(x, \gamma(x))$ is asymptotically stable, so is the origin of $(\star)$
- If the origin of $\dot{x} = f(x, \gamma(x))$ is exponentially stable, so is the origin of $(\star)$
- If the assumptions hold globally and the system $\dot{x} = f(x, \gamma(x - \tilde{x}))$, with input $\tilde{x}$, is ISS, then the origin of $(\star)$ is globally asymptotically stable

High-Gain Observers

Example 12.3

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = \phi(x,u), \quad y = x_1$$

State feedback control: $u = \gamma(x)$ stabilizes the origin of

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = \phi(x,\gamma(x))$$

High-gain observer

$$\dot{\hat{x}}_1 = \hat{x}_2 + (\alpha_1/\varepsilon)(y - \hat{x}_1), \quad \dot{\hat{x}}_2 = \phi_0(\hat{x},u) + (\alpha_2/\varepsilon^2)(y - \hat{x}_1)$$

ϕ_0 is a nominal model of ϕ, $\alpha_i > 0$, $0 < \varepsilon \ll 1$

$$|\tilde{x}_1| \leq \max\left\{be^{-at/\varepsilon},\ \varepsilon^2 cM\right\}, \quad |\tilde{x}_2| \leq \left\{\frac{b}{\varepsilon}e^{-at/\varepsilon},\ \varepsilon cM\right\}$$

The bound on $\tilde{x}_2$ demonstrates the peaking phenomenon, which might destabilize the closed-loop system

Example:

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = x_2^3 + u, \qquad y = x_1$$

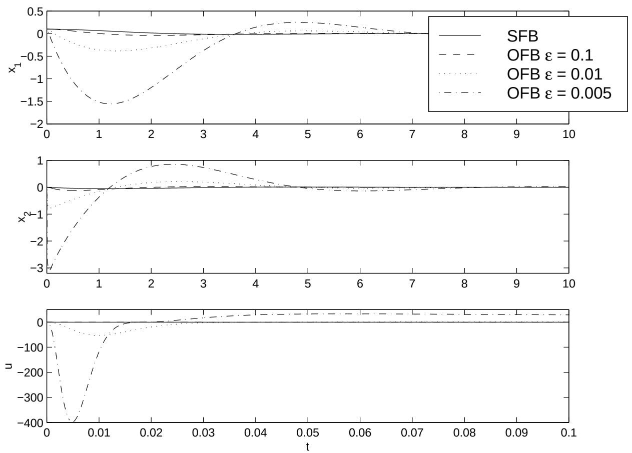
State feedback control:

$$u = -x_2^3 - x_1 - x_2$$

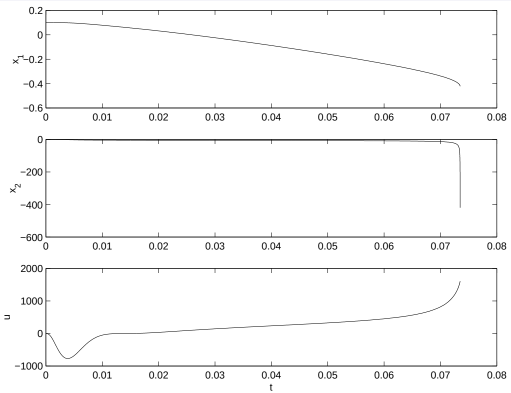
Output feedback control:

$$u = -\hat{x}_2^3 - \hat{x}_1 - \hat{x}_2$$

$$\dot{\hat{x}}_1 = \hat{x}_2 + (2/\varepsilon)(y - \hat{x}_1), \quad \dot{\hat{x}}_2 = (1/\varepsilon^2)(y - \hat{x}_1)$$


SFB
OFB ε = 0.1
OFB ε = 0.01
OFB ε = 0.005
x1
x2
u
t

$\varepsilon = 0.004$



Closed-loop system under state feedback:

$$\dot{x} = Ax, \qquad A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

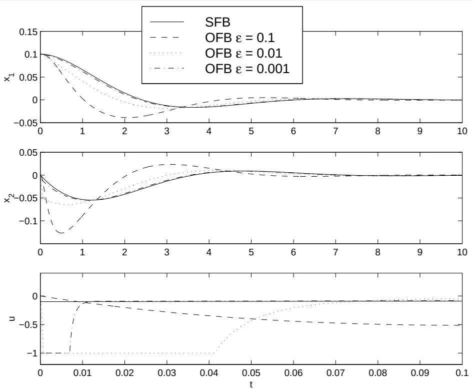
$$PA + A^T P = -I \qquad \Rightarrow \qquad P = \begin{bmatrix} 1.5 & 0.5 \\ 0.5 & 1 \end{bmatrix}$$

Suppose $x(0)$ belongs to the positively invariant set $\Omega = \{V(x) \le 0.3\}$

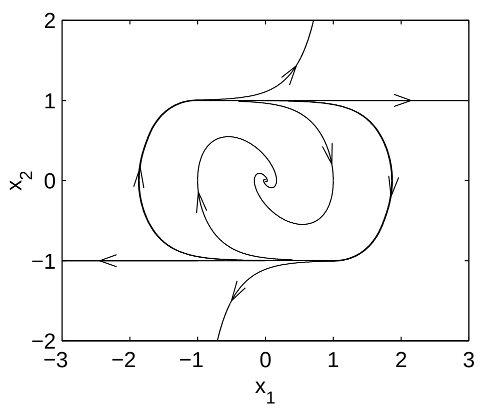
$$|u| \le |x_2|^3 + |x_1 + x_2| \le 0.816, \quad \forall \, x \in \Omega$$

Saturate u at ± 1

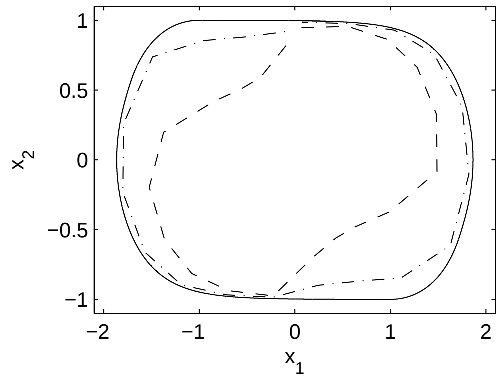
$$u = \mathrm{sat}(-\hat{x}_2^3 - \hat{x}_1 - \hat{x}_2)$$



Region of attraction under state feedback:



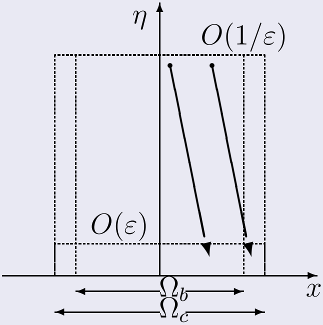
Region of attraction under output feedback:



$\varepsilon = 0.08$ (dashed) and $\varepsilon = 0.01$ (dash-dot)

Analysis of the closed-loop system:

$$\begin{aligned} \dot{x}_1 &= x_2 & \qquad \dot{x}_2 &= \phi(x, \gamma(x - \tilde{x})) \\ \varepsilon\dot{\eta}_1 &= -\alpha_1\eta_1 + \eta_2 & \qquad \varepsilon\dot{\eta}_2 &= -\alpha_2\eta_1 + \varepsilon\delta(x, \tilde{x}) \end{aligned}$$



General case

$$
\begin{aligned}
\dot{w} &= \psi(w, x, u) \\
\dot{x}_i &= x_{i+1} + \psi_i(x_1, \dots, x_i, u), \qquad 1 \le i \le \rho - 1 \\
\dot{x}_\rho &= \phi(w, x, u) \\
y &= x_1 \\
z &= q(w, x)
\end{aligned}
$$

$$\phi(0,0,0) = 0, \quad \psi(0,0,0) = 0, \quad q(0,0) = 0$$

The normal form and models of mechanical and electromechanical systems take this form with

$$\psi_1 = \cdots = \psi_\rho = 0$$

Why the extra measurement z?

Stabilizing state feedback controller:

$$\dot{\vartheta} = \Gamma(\vartheta, x, z), \qquad u = \gamma(\vartheta, x, z)$$

γ and Γ are globally bounded functions of x

Closed-loop system

$$\dot{\mathcal{X}} = f(\mathcal{X}), \quad \mathcal{X} = \mathrm{col}(w, x, \vartheta)$$

Output feedback controller

$$\dot{\vartheta} = \Gamma(\vartheta, \hat{x}, z), \quad u = \gamma(\vartheta, \hat{x}, z)$$

Observer

$$\begin{aligned}
\dot{\hat{x}}_i &= \hat{x}_{i+1} + \psi_i(\hat{x}_1, \ldots, \hat{x}_i, u) + \frac{\alpha_i}{\varepsilon^i}(y - \hat{x}_1), \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 1 \le i \le \rho - 1 \\
\dot{\hat{x}}_\rho &= \phi_0(z, \hat{x}, u) + \frac{\alpha_\rho}{\varepsilon^\rho}(y - \hat{x}_1)
\end{aligned}$$

$\varepsilon > 0$ and α_1 to α_ρ are chosen such that the roots of

$$s^\rho + \alpha_1 s^{\rho-1} + \cdots + \alpha_{\rho-1} s + \alpha_\rho = 0$$

have negative real parts

Separation Principle

Theorem 12.2

Suppose the origin of $\dot{\mathcal{X}} = f(\mathcal{X})$ is asymptotically stable and $\mathcal{R}$ is its region of attraction. Let $\mathcal{S}$ be any compact set in the interior of $\mathcal{R}$ and $\mathcal{Q}$ be any compact subset of R^ρ. Then, given any $\mu > 0$ there exist $\varepsilon^* > 0$ and $T^* > 0$, dependent on μ, such that for every $0 < \varepsilon \leq \varepsilon^*$, the solutions $(\mathcal{X}(t), \hat{x}(t))$ of the closed-loop system, starting in $\mathcal{S} \times \mathcal{Q}$, are bounded for all $t \geq 0$ and satisfy

$$\|\mathcal{X}(t)\| \leq \mu \quad \text{and} \quad \|\hat{x}(t)\| \leq \mu, \quad \forall\ t \geq T^*$$

$$\|\mathcal{X}(t) - \mathcal{X}_r(t)\| \leq \mu, \quad \forall\ t \geq 0$$

where $\mathcal{X}_r$ is the solution of $\dot{\mathcal{X}} = f(\mathcal{X})$, starting at $\mathcal{X}(0)$

If the origin of $\dot{\mathcal{X}} = f(\mathcal{X})$ is exponentially stable, then the origin of the closed-loop system is exponentially stable and $\mathcal{S} \times \mathcal{Q}$ is a subset of its region of attraction

Robust Stabilization of Minimum Phase Systems

Relative Degree One

$$\dot{\eta} = f_0(\eta, y), \qquad \dot{y} = a(\eta, y) + b(\eta, y)u + \delta(t, \eta, y, u)$$

$$f_0(0,0) = 0, \quad a(0,0) = 0, \quad b(\eta, y) \geq b_0 > 0$$

The origin of $\dot{\eta} = f_0(\eta, 0)$ is asymptotically stable

$$\alpha_1(\|\eta\|) \leq V(\eta) \leq \alpha_2(\|\eta\|)$$

$$\frac{\partial V}{\partial \eta} f_0(\eta, y) \leq -\alpha_3(\|\eta\|), \qquad \forall \ \|\eta\| \geq \alpha_4(|y|)$$

Sliding Mode Control: Sliding surface $y = 0$

$$u = \psi(y) + v$$

$$\left|\frac{a(\eta, y) + b(\eta, y)\psi(y) + \delta(t, \eta, y, \psi(y) + v)}{b(\eta, y)}\right| \le \varrho(y) + \kappa_0|v|$$

$$0 \le \kappa_0 < 1$$

$$\beta(y) \ge \frac{\varrho(y)}{1 - \kappa_0} + \beta_0$$

$$v = -\beta(y) \text{ sat}\left(\frac{y}{\mu}\right)$$

$$u = \psi(y) - \beta(y) \text{ sat}\left(\frac{y}{\mu}\right)$$

All the assumptions hold in a domain D

Relative Degree Higher Than One

$$\begin{aligned}
\dot{\eta} &= f_0(\eta, \xi) \\
\dot{\xi}_i &= \xi_{i+1}, \quad \text{for } 1 \leq i \leq \rho - 1 \\
\dot{\xi}_\rho &= a(\eta, \xi) + b(\eta, \xi)u + \delta(t, \eta, \xi, u) \\
y &= \xi_1
\end{aligned}$$

$$f_0(0,0) = 0, \quad a(0,0) = 0, \quad b(\eta, \xi) \geq b_0 > 0$$

The origin of $\dot{\eta} = f_0(\eta, 0)$ is asymptotically stable

Partial State Feedback: Assume ξ is available for feedback

$$s = k_1\xi_1 + k_2\xi_2 + \cdots + k_{\rho-1}\xi_{\rho-1} + \xi_\rho$$

With s as the output, the system has relative degree one and the normal form is given by

$$\dot{z} = \bar{f}_0(z, s), \qquad \dot{s} = \bar{a}(z, s) + \bar{b}(z, s)u + \bar{\delta}(t, z, s, u)$$

$$z = \mathrm{col}\left(\eta, \quad \xi_1, \quad \ldots \quad \xi_{\rho-2}, \quad \xi_{\rho-1}\right)$$

Zero Dynamics ($s = 0$):

$$\dot{z} = \bar{f}_0(z, 0)$$

$$\dot{z} = \bar{f}_0(z, 0)$$

$$\Leftrightarrow \quad \dot{\eta} = \left. f_0(\eta, \xi) \right|_{\xi_\rho = -\sum_{i=1}^{\rho-1} k_i \xi_i}, \quad \dot{\zeta} = F\zeta$$

$$\zeta = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_{\rho-1} \end{bmatrix}, \qquad F = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & & 0 & 1 \\ -k_1 & -k_2 & \cdots & -k_{\rho-2} & -k_{\rho-1} \end{bmatrix}$$

When $\rho = n$, the zero dynamics are $\dot{\zeta} = F\zeta$

k_1 to $k_{\rho-1}$ are chosen such that the polynomial

$$\lambda^{\rho-1} + k_{\rho-1}\lambda^{\rho-2} + \cdots + k_2\lambda + k_1$$

is Hurwitz

$$\alpha_1(\|z\|) \le V(z) \le \alpha_2(\|z\|)$$

$$\frac{\partial V}{\partial \eta}\bar{f}_0(z,s) \le -\alpha_3(\|z\|), \qquad \forall\ \|z\| \ge \alpha_4(|s|)$$

We have converted the relative degree ρ system into a relative degree one system that satisfies the earlier assumptions

$$u = \psi(\xi) + v$$

$$\left|\frac{\bar{a}(z,s)+\bar{b}(z,s)\psi(\xi)+\bar{\delta}(t,z,s,\psi(\xi)+v)}{\bar{b}(z,s)}\right| \leq \varrho(\xi)+\kappa_0|v|$$

Left hand side equals

$$\left|\frac{\sum_{i=1}^{\rho-1} k_i\xi_{i+1}+a(\eta,\xi)+b(\eta,\xi)\psi(\xi)+\delta(t,\eta,\xi,\psi(\xi)+v)}{b(\eta,\xi)}\right|$$

$$\beta(\xi) \geq \frac{\varrho(\xi)}{1-\kappa_0}+\beta_0, \quad \beta_0 > 0$$

$$u = \psi(\xi) - \beta(\xi)\ \mathrm{sat}\left(\frac{s}{\mu}\right)$$

Saturate β and ψ