Nonlinear Control

Lecture # 14

Tracking & Regulation

Normal form:

$$\begin{aligned}
\dot{\eta} &= f_0(\eta, \xi) \\
\dot{\xi}_i &= \xi_{i+1}, \qquad \text{for } 1 \le i \le \rho - 1 \\
\dot{\xi}_\rho &= a(\eta, \xi) + b(\eta, \xi)u \\
y &= \xi_1
\end{aligned}$$

$$\eta \in D_\eta \subset R^{n-\rho},\ \xi = \mathrm{col}(\xi_1, \ldots, \xi_\rho) \in D_\xi \subset R^\rho$$

Tracking Problem: Design a feedback controller such that

$$\lim_{t\to\infty} [y(t) - r(t)] = 0$$

while ensuring boundedness of all state variables

Regulation Problem: r is constant

Assumption 13.1

$$b(\eta, \xi) \geq b_0 > 0, \qquad \forall\ \eta \in D_\eta,\ \xi \in D_\xi$$

Assumption 13.2

$\dot{\eta} = f_0(\eta, \xi)$ is bounded-input–bounded-state stable over $D_\eta \times D_\xi$

Assumption 13.2 holds locally if the system is minimum phase and globally if $\dot{\eta} = f_0(\eta, \xi)$ is ISS

Assumption 13.3

$r(t)$ and its derivatives up to $r^{(\rho)}(t)$ are bounded for all $t \geq 0$ and the ρth derivative $r^{(\rho)}(t)$ is a piecewise continuous function of t. Moreover, $\mathcal{R} = \mathrm{col}(r, \dot{r}, \ldots, r^{(\rho-1)}) \in D_\xi$ for all $t \geq 0$

The reference signal $r(t)$ could be specified as given functions of time, or it could be the output of a reference model

Example: For $\rho = 2$

$$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \quad \zeta > 0,\ \omega_n > 0$$

$$\dot{y}_1 = y_2, \qquad \dot{y}_2 = -\omega_n^2 y_1 - 2\zeta\omega_n y_2 + \omega_n^2 u_c, \qquad r = y_1$$

$$\dot{r} = y_2, \quad \ddot{r} = \dot{y}_2$$

Assumption 13.3 is satisfied when $u_c(t)$ is piecewise continuous and bounded

Change of variables:

$$e_1 = \xi_1 - r, \quad e_2 = \xi_2 - r^{(1)}, \quad \ldots, \quad e_\rho = \xi_\rho - r^{(\rho-1)}$$

$$\begin{aligned} \dot{\eta} &= f_0(\eta, \xi) \\ \dot{e}_i &= e_{i+1}, \qquad \text{for } 1 \le i \le \rho - 1 \\ \dot{e}_\rho &= a(\eta, \xi) + b(\eta, \xi)u - r^{(\rho)} \end{aligned}$$

Goal: Ensure $e = \mathrm{col}(e_1, \ldots, e_\rho) = \xi - \mathcal{R}$ is bounded for all $t \ge 0$ and converges to zero as t tends to infinity

Assumption 13.4

$r,\ r^{(1)}, \ldots,\ r^{(\rho)}$ are available to the controller (needed in state feedback control)

Feedback controllers for tracking and regulation are classified as in stabilization

- State versus output feedback
- Static versus dynamic controllers
- Region of validity
  - local tracking
  - regional tracking
  - semiglobal tracking
  - global tracking

Local tracking is achieved for sufficiently small initial states and sufficiently small $\|\mathcal{R}\|$, while global tracking is achieved for any initial state and any bounded $\mathcal{R}$.

Practical tracking: The tracking error is ultimately bounded and the ultimate bound can be made arbitrarily small by choice of design parameters

- local practical tracking
- regional practical tracking
- semiglobal practical tracking
- global practical tracking

Tracking

$$\dot{\eta} = f_0(\eta, \xi), \qquad \dot{e} = A_c e + B_c \left[a(\eta, \xi) + b(\eta, \xi)u - r^{(\rho)}\right]$$

Feedback linearization:

$$u = \left[-a(\eta, \xi) + r^{(\rho)} + v\right] / b(\eta, \xi)$$

$$\dot{\eta} = f_0(\eta, \xi), \qquad \dot{e} = A_c e + B_c v$$

$v = -Ke$, $\quad A_c - B_c K$ is Hurwitz

$$\dot{\eta} = f_0(\eta, \xi), \quad \dot{e} = (A_c - B_c K)e$$

$A_c - B_c K$ Hurwitz $\Rightarrow$ $e(t)$ is bounded and $\lim_{t\to\infty} e(t) = 0$

$\Rightarrow$ $\xi = e + \mathcal{R}$ is bounded $\Rightarrow$ η is bounded

Example 13.1 (Pendulum equation)

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\sin x_1 - bx_2 + cu, \qquad y = x_1$$

We want the output y to track a reference signal $r(t)$

$$e_1 = x_1 - r, \qquad e_2 = x_2 - \dot{r}$$

$$\dot{e}_1 = e_2, \qquad \dot{e}_2 = -\sin x_1 - bx_2 + cu - \ddot{r}$$

$$u = \frac{1}{c}[\sin x_1 + bx_2 + \ddot{r} - k_1 e_1 - k_2 e_2]$$

$K = [k_1, k_2]$ assigns the eigenvalues of $A_c - B_c K$ at desired locations in the open left-half complex plane

Simulation

$$r = \sin(t/3), \quad x(0) = \mathrm{col}(\pi/2, 0)$$

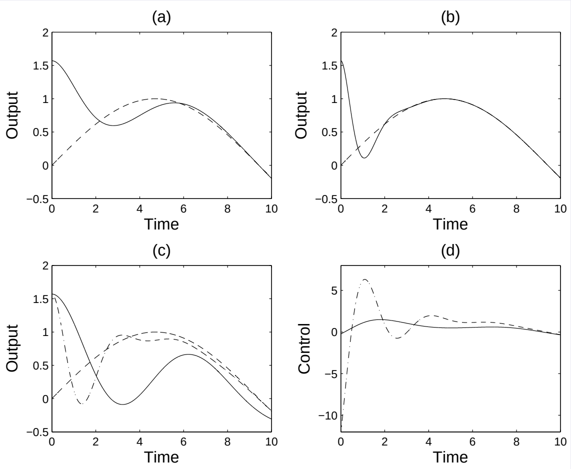
Nominal: $b = 0.03, c = 1$ Figures (a) and (b)

Perturbed: $b = 0.015, c = 0.5$ Figure (c)

Reference (dashed)

Low gain: $K = \begin{bmatrix} 1 & 1 \end{bmatrix}$, $\lambda = -0.5 \pm j0.5\sqrt{3}$, (solid)

High gain: $K = \begin{bmatrix} 9 & 3 \end{bmatrix}$, $\lambda = -1.5 \pm j1.5\sqrt{3}$, (dash-dot)


(a)
(b)
(c)
(d)
Output
Control
Time
2
1.5
1
0.5
0
−0.5
5
−5
−10
0
2
4
6
8
10

Robust Tracking

$$
\begin{aligned}
\dot{\eta} &= f_0(\eta, \xi) \\
\dot{e}_i &= e_{i+1}, \qquad 1 \le i \le \rho - 1 \\
\dot{e}_\rho &= a(\eta, \xi) + b(\eta, \xi)u + \delta(t, \eta, \xi, u) - r^{(\rho)}(t)
\end{aligned}
$$

Sliding mode control: Design the sliding surface

$$\dot{e}_i = e_{i+1}, \qquad 1 \le i \le \rho - 1$$

View e_ρ as the control input and design it to stabilize the origin

$$e_\rho = -(k_1 e_1 + \cdots + k_{\rho-1} e_{\rho-1})$$

$$\lambda^{\rho-1} + k_{\rho-1}\lambda^{\rho-2} + \cdots + k_1 \text{ is Hurwitz}$$

$$s = (k_1 e_1 + \cdots + k_{\rho-1} e_{\rho-1}) + e_\rho = 0$$

$$\dot{s} = \sum_{i=1}^{\rho-1} k_i e_{i+1} + a(\eta,\xi) + b(\eta,\xi)u + \delta(t,\eta,\xi,u) - r^{(\rho)}(t)$$

$$u = v \text{ or } u = -\frac{1}{\hat{b}(\eta,\xi)} \left[\sum_{i=1}^{\rho-1} k_i e_{i+1} + \hat{a}(\eta,\xi) - r^{(\rho)}(t) \right] + v$$

$$\dot{s} = b(\eta,\xi)v + \Delta(t,\eta,\xi,v)$$

Suppose $\left| \dfrac{\Delta(t,\eta,\xi,v)}{b(\eta,\xi)} \right| \leq \varrho(\eta,\xi) + \kappa_0 |v|, \quad 0 \leq \kappa_0 < 1$

$$v = -\beta(\eta,\xi) \operatorname{sat}\left(\frac{s}{\mu}\right), \quad \beta(\eta,\xi) \geq \frac{\varrho(\eta,\xi)}{(1-\kappa_0)} + \beta_0, \ \beta_0 > 0$$

$$s\dot{s} \le -\beta_0 b_0 (1-\kappa_0)|s|, \quad |s| \ge \mu$$

$$\zeta = \mathrm{col}(e_1, \ldots, e_{\rho-1}), \quad \dot{\zeta} = \underbrace{(A_c - B_c K)}_{\text{Hurwitz}} \zeta + B_c s$$

$$V_0 = \zeta^T P \zeta, \quad P(A_c - B_c K) + (A_c - B_c K)^T P = -I$$

$$\dot{V}_0 = -\zeta^T\zeta + 2\zeta^T P B_c s \le -(1-\theta)\|\zeta\|^2, \ \forall \ \|\zeta\| \ge 2\|PB_c\| \ |s|/\theta$$

$0 < \theta < 1$. For $\sigma \ge \mu$

$$\{\|\zeta\| \le 2\|PB_c\| \ \sigma/\theta\} \subset \{\zeta^T P \zeta \le \lambda_{\max}(P)(2\|PB_c\|/\theta)^2\sigma^2\}$$

$$\rho_1 = \lambda_{\max}(P)(2\|PB_c\|/\theta)^2, \quad c > \mu$$

$\Omega = \{\zeta^T P \zeta \le \rho_1 c^2\} \times \{|s| \le c\}$ is positively invariant

For all $e(0) \in \Omega$, $e(t)$ enters the positively invariant set

$$\Omega_\mu = \{\zeta^T P \zeta \leq \rho_1 \mu^2\} \times \{|s| \leq \mu\}$$

Inside Ω_μ,

$$|e_1| \leq k\mu$$

$$k = \|LP^{-1/2}\| \sqrt{\rho_1}, \quad L = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}$$

Example 13.2 (Reconsider Example 13.1)

$$\dot{e}_1 = e_2, \qquad \dot{e}_2 = -\sin x_1 - bx_2 + cu - \ddot{r}$$

$$r(t) = \sin(t/3), \quad 0 \le b \le 0.1, \quad 0.5 \le c \le 2$$

$$s = e_1 + e_2$$

$$\dot{s} = e_2 - \sin x_1 - bx_2 + cu - \ddot{r} = (1-b)e_2 - \sin x_1 - b\dot{r} - \ddot{r}$$

$$\left| \frac{(1-b)e_2 - \sin x_1 - b\dot{r} - \ddot{r}}{c} \right| \le \frac{|e_2| + 1 + 0.1/3 + 1/9}{0.5}$$

$$u = -(2|e_2| + 3)\ \mathrm{sat}\left(\frac{e_1 + e_2}{\mu}\right)$$

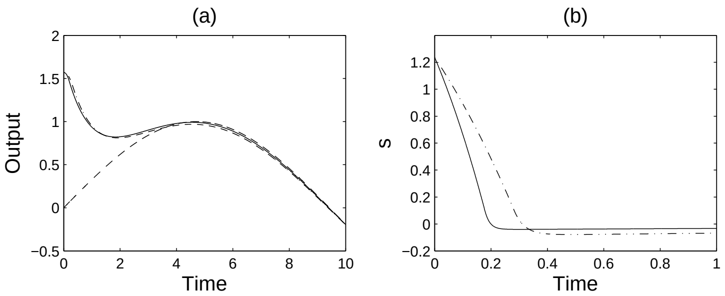
Simulation:

$$\mu = 0.1, \quad x(0) = \mathrm{col}(\pi/2, 0)$$

$b = 0.03$, $c = 1$ (solid)

$b = 0.015$, $c = 0.5$ (dash-dot)

Reference (dashed)


(a)
Output
Time
2
1.5
1
0.5
0
−0.5
0
2
4
6
8
10
(b)
s
Time
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
0
0.2
0.4
0.6
0.8
1

Robust Regulation via Integral Action

$$
\begin{aligned}
\dot{\eta} &= f_0(\eta, \xi, w) \\
\dot{\xi}_i &= \xi_{i+1}, \qquad 1 \le i \le \rho - 1 \\
\dot{\xi}_\rho &= a(\eta, \xi, w) + b(\eta, \xi, w)u \\
y &= \xi_1
\end{aligned}
$$

Disturbance w and reference r are constant

Equilibrium point:

$$
\begin{aligned}
0 &= f_0(\bar{\eta}, \bar{\xi}, w) \\
0 &= \bar{\xi}_{i+1}, \qquad 1 \le i \le \rho - 1 \\
0 &= a(\bar{\eta}, \bar{\xi}, w) + b(\bar{\eta}, \bar{\xi}, w)\bar{u} \\
r &= \bar{\xi}_1
\end{aligned}
$$

Assumption 13.5

$0 = f_0(\bar{\eta}, \bar{\xi}, w)$ has a unique solution $\bar{\eta} = \phi_\eta(r, w)$

$$\bar{u} = -\,\frac{a(\bar{\eta}, \bar{\xi}, w)}{b(\bar{\eta}, \bar{\xi}, w)} \stackrel{\text{def}}{=} \phi_u(r, w)$$

Augment the integrator $\quad \dot{e}_0 = y - r$

$$z = \eta - \bar{\eta}, \qquad e = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_\rho \end{bmatrix} = \begin{bmatrix} \xi_1 - r \\ \xi_2 \\ \vdots \\ \xi_\rho \end{bmatrix}$$

$$\begin{aligned}\dot{z} &= f_0(z+\bar{\eta}, \xi, w) \stackrel{\text{def}}{=} \tilde{f}_0(z, e, r, w) \\ \dot{e}_i &= e_{i+1}, \qquad \text{for } 0 \le i \le \rho - 1 \\ \dot{e}_\rho &= a(\eta, \xi, w) + b(\eta, \xi, w)u\end{aligned}$$

Sliding mode control:

$$s = k_0 e_0 + k_1 e_1 + \cdots + k_{\rho-1} e_{\rho-1} + e_\rho$$

$$\lambda^\rho + k_{\rho-1}\lambda^{\rho-1} + \cdots + k_1\lambda + k_0 \ \text{ is Hurwitz}$$

$$\dot{s} = \sum_{i=0}^{\rho-1} k_i e_{i+1} + a(\eta, \xi, w) + b(\eta, \xi, w)u$$

$$u = v \quad \text{or} \quad u = -\frac{1}{\hat{b}(\eta, \xi)}\left[\sum_{i=0}^{\rho-1} k_i e_{i+1} + \hat{a}(\eta, \xi)\right] + v$$

$$\dot{s} = b(\eta, \xi, w)v + \Delta(\eta, \xi, r, w)$$

$$\left|\frac{\Delta(\eta, \xi, r, w)}{b(\eta, \xi, w)}\right| \leq \varrho(\eta, \xi)$$

$$v = -\beta(\eta, \xi)\ \mathrm{sat}\left(\frac{s}{\mu}\right), \quad \beta(\eta, \xi) \geq \varrho(\eta, \xi) + \beta_0,\ \beta_0 > 0$$

Assumption 13.6

$$\alpha_1(\|z\|) \leq V_1(z, r, w) \leq \alpha_2(\|z\|)$$

$$\frac{\partial V_1}{\partial z}\tilde{f}_0(z, e, r, w) \leq -\alpha_3(\|z\|), \qquad \forall\ \|z\| \geq \alpha_4(\|e\|)$$

Assumption 13.7

$z = 0$ is an exponentially stable equilibrium point of
$\dot{z} = \tilde{f}_0(z, 0, r, w)$

Theorem 13.1

Under the stated assumptions, there are positive constants c, ρ_1 and ρ_2 and a positive definite matrix P such that the set

$$\Omega = \{V_1(z) \leq \alpha_2(\alpha_4(c\rho_2))\} \times \{\zeta^T P \zeta \leq \rho_1 c^2\} \times \{|s| \leq c\}$$

where $\zeta = \mathrm{col}(e_0, e_1, \ldots, e_{\rho-1})$, is compact and positively invariant, and for all initial states in Ω

$$\lim_{t\to\infty} |y(t) - r| = 0$$

Special case: $\beta = k$ (a constant) and $u = v$

$$u = -k \ \mathrm{sat}\left(\frac{k_0 e_0 + k_1 e_1 + \cdots + k_{\rho-1} e_{\rho-1} + e_\rho}{\mu}\right)$$

Example 13.4 (Pendulum with horizontal acceleration)

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\sin x_1 - bx_2 + cu + d\cos x_1, \quad y = x_1$$

d is constant. Regulate y to a constant reference r

$$0 \le b \le 0.1, \quad 0.5 \le c \le 2, \quad 0 \le d \le 0.5$$

$$e_1 = x_1 - r, \quad e_2 = x_2$$

$$\dot{e}_0 = e_1, \quad \dot{e}_1 = e_2, \quad \dot{e}_2 = -\sin x_1 - bx_2 + cu + d\cos x_1$$

$$s = e_0 + 2e_1 + e_2$$

$$\dot{s} = e_1 + (2-b)e_2 - \sin x_1 + cu + d\cos x_1$$

$$\left|\frac{e_1 + (2-b)e_2 - \sin x_1 + d\cos x_1}{c}\right| \le \frac{|e_1| + 2|e_2| + 1 + 0.5}{0.5}$$

$$u = -(2|e_1| + 4|e_2| + 4)\ \mathrm{sat}\left(\frac{e_0 + 2e_1 + e_2}{\mu}\right)$$

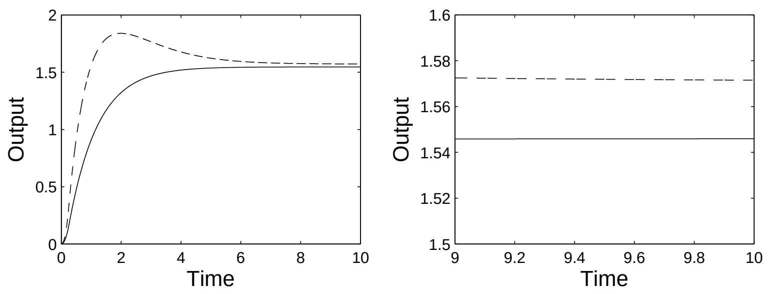
For comparison, SMC without integrator

$$s = e_1 + e_2, \quad \dot{s} = (1-b)e_2 - \sin x_1 + cu + d\cos x_1$$

$$u = -(2|e_2| + 4)\ \mathrm{sat}\left(\frac{e_1 + e_2}{\mu}\right)$$

Simulation: With integrator (dashed), without (solid)

$$\mu = 0.1,\ x(0) = 0,\ r = \pi/2,\ b = 0.03,\ c = 1,\ d = 0.3$$



2
1.5
1
0.5
0
Output
0
2
4
6
8
10
Time
1.6
1.58
1.56
1.54
1.52
1.5
Output
9
9.2
9.4
9.6
9.8
10
Time

Output Feedback

Tracking:

$$
\begin{aligned}
\dot{\eta} &= f_0(\eta, \xi) \\
\dot{e}_i &= e_{i+1}, \qquad 1 \le i \le \rho - 1 \\
\dot{e}_\rho &= a(\eta, \xi) + b(\eta, \xi)u + \delta(t, \eta, \xi, u) - r^{(\rho)}(t)
\end{aligned}
$$

Regulation:

$$
\begin{aligned}
\dot{\eta} &= f_0(\eta, \xi, w) \\
\dot{\xi}_i &= \xi_{i+1}, \qquad 1 \le i \le \rho - 1 \\
\dot{\xi}_\rho &= a(\eta, \xi, w) + b(\eta, \xi, w)u \\
y &= \xi_1
\end{aligned}
$$

- Design partial state feedback control that uses ξ
- Use a high-gain observer

Tracking sliding mode controller:

$$u = -\beta(\xi)\ \mathrm{sat}\left(\frac{k_1 e_1 + \cdots + k_{\rho-1} e_{\rho-1} + e_\rho}{\mu}\right)$$

Regulation sliding mode controller:

$$u = -\beta(\xi)\ \mathrm{sat}\left(\frac{k_0 e_0 + k_1 e_1 + \cdots + k_{\rho-1} e_{\rho-1} + e_\rho}{\mu}\right)$$

$$\dot{e}_0 = e_1 = y - r$$

β is allowed to depend only on ξ rather than the full state vector. On compact sets, the η-dependent part of $\varrho(\eta, \xi)$ can be bounded by a constant

High-gain observer:

$$\begin{aligned}\dot{\hat{e}}_i &= \hat{e}_{i+1} + \frac{\alpha_i}{\varepsilon^i}(y - r - \hat{e}_1), \quad 1 \leq i \leq \rho - 1 \\ \dot{\hat{e}}_\rho &= \frac{\alpha_\rho}{\varepsilon^\rho}(y - r - \hat{e}_1)\end{aligned}$$

$$\lambda^\rho + \alpha_1\lambda^{\rho-1} + \cdots + \alpha_{\rho-1}\lambda + \alpha_\rho \text{ Hurwitz}$$

$$e \;\rightarrow \hat{e}$$

$$\xi \;\rightarrow \hat{\xi} = \hat{e} + \mathcal{R}$$

$$\beta(\hat{\xi}) \;\rightarrow\; \beta_s(\hat{\xi}) \text{ (saturated)}$$

Tracking:

$$u = -\beta_s(\hat{\xi})\ \mathrm{sat}\left(\frac{k_1\hat{e}_1 + \cdots + k_{\rho-1}\hat{e}_{\rho-1} + \hat{e}_\rho}{\mu}\right)$$

Regulation:

$$u = -\beta_s(\hat{\xi})\ \mathrm{sat}\left(\frac{k_0 e_0 + k_1\hat{e}_1 + \cdots + k_{\rho-1}\hat{e}_{\rho-1} + \hat{e}_\rho}{\mu}\right)$$

We can replace $\hat{e}_1$ by e_1

Special case: When β_s is constant or function of $\hat{e}$ rather than $\hat{\xi}$, we do not need the derivatives of r, as required by Assumption 13.4

The output feedback controllers recover the performance of the partial state feedback controllers for sufficiently small ε. In the regulation case, the regulation error converges to zero

Relative degree one systems: No observer

$$u = -\beta(y)\,\mathrm{sat}\left(\frac{y-r}{\mu}\right), \qquad u = -\beta(y)\,\mathrm{sat}\left(\frac{k_0 e_0 + y - r}{\mu}\right)$$