

Nonlinear Control

Lecture # 3

Stability of Equilibrium Points

The Invariance Principle

Definitions

Let $x(t)$ be a solution of $\dot{x} = f(x)$

A point p is a **positive limit point** of $x(t)$ if there is a sequence $\{t_n\}$, with $\lim_{n \rightarrow \infty} t_n = \infty$, such that $x(t_n) \rightarrow p$ as $n \rightarrow \infty$

The set of all positive limit points of $x(t)$ is called the **positive limit set** of $x(t)$; denoted by L^+

If $x(t)$ approaches an asymptotically stable equilibrium point $\bar{x}$, then $\bar{x}$ is the positive limit point of $x(t)$ and $L^+ = \bar{x}$

A stable limit cycle is the positive limit set of every solution starting sufficiently near the limit cycle

A set M is an *invariant set* with respect to $\dot{x} = f(x)$ if

$$x(0) \in M \Rightarrow x(t) \in M, \quad \forall t \in R$$

Examples:

- Equilibrium points
- Limit Cycles

A set M is a *positively invariant set* with respect to $\dot{x} = f(x)$ if

$$x(0) \in M \Rightarrow x(t) \in M, \quad \forall t \geq 0$$

Example; The set $\Omega_c = \{V(x) \leq c\}$ with $\dot{V}(x) \leq 0$ in Ω_c

The distance from a point p to a set M is defined by

$$\text{dist}(p, M) = \inf_{x \in M} \|p - x\|$$

$x(t)$ approaches a set M as t approaches infinity, if for each $\varepsilon > 0$ there is $T > 0$ such that

$$\text{dist}(x(t), M) < \varepsilon, \quad \forall t > T$$

Example: every solution $x(t)$ starting sufficiently near a stable limit cycle approaches the limit cycle as $t \rightarrow \infty$

Notice, however, that $x(t)$ does not converge to any specific point on the limit cycle

Lemma 3.1

If a solution $x(t)$ of $\dot{x} = f(x)$ is bounded and belongs to D for $t \geq 0$, then its positive limit set L^+ is a nonempty, compact, invariant set. Moreover, $x(t)$ approaches L^+ as $t \rightarrow \infty$

LaSalle's Theorem (3.4)

Let $f(x)$ be a locally Lipschitz function defined over a domain $D \subset \mathbb{R}^n$ and $\Omega \subset D$ be a compact set that is positively invariant with respect to $\dot{x} = f(x)$. Let $V(x)$ be a continuously differentiable function defined over D such that $\dot{V}(x) \leq 0$ in Ω . Let E be the set of all points in Ω where $\dot{V}(x) = 0$, and M be the largest invariant set in E . Then every solution starting in Ω approaches M as $t \rightarrow \infty$

Proof

$$\dot{V}(x) \leq 0 \text{ in } \Omega \Rightarrow V(x(t)) \text{ is a decreasing}$$

$$V(x) \text{ is continuous in } \Omega \Rightarrow V(x) \geq b = \min_{x \in \Omega} V(x)$$

$$\Rightarrow \lim_{t \rightarrow \infty} V(x(t)) = a$$

$$x(t) \in \Omega \Rightarrow x(t) \text{ is bounded} \Rightarrow L^+ \text{ exists}$$

Moreover, $L^+ \subset \Omega$ and $x(t)$ approaches L^+ as $t \rightarrow \infty$

For any $p \in L^+$, there is $\{t_n\}$ with $\lim_{n \rightarrow \infty} t_n = \infty$ such that $x(t_n) \rightarrow p$ as $n \rightarrow \infty$

$$V(x) \text{ is continuous} \Rightarrow V(p) = \lim_{n \rightarrow \infty} V(x(t_n)) = a$$

$$V(x) = a \text{ on } L^+ \text{ and } L^+ \text{ invariant} \Rightarrow \dot{V}(x) = 0, \forall x \in L^+$$

$$L^+ \subset M \subset E \subset \Omega$$

$$x(t) \text{ approaches } L^+ \Rightarrow x(t) \text{ approaches } M \text{ (as } t \rightarrow \infty)$$

Theorem 3.5

Let $f(x)$ be a locally Lipschitz function defined over a domain $D \subset \mathbb{R}^n$; $0 \in D$. Let $V(x)$ be a continuously differentiable positive definite function defined over D such that $\dot{V}(x) \leq 0$ in D . Let $S = \{x \in D \mid \dot{V}(x) = 0\}$

- If no solution can stay identically in S , other than the trivial solution $x(t) \equiv 0$, then the origin is asymptotically stable
- Moreover, if $\Gamma \subset D$ is compact and positively invariant, then it is a subset of the region of attraction
- Furthermore, if $D = \mathbb{R}^n$ and $V(x)$ is radially unbounded, then the origin is globally asymptotically stable

Example 3.8

$$\begin{aligned}\dot{x}_1 &= x_2, & \dot{x}_2 &= -h_1(x_1) - h_2(x_2) \\ h_i(0) &= 0, & y h_i(y) &> 0, \text{ for } 0 < |y| < a\end{aligned}$$

$$V(x) = \int_0^{x_1} h_1(y) dy + \frac{1}{2}x_2^2$$

$$D = \{-a < x_1 < a, \quad -a < x_2 < a\}$$

$$\dot{V}(x) = h_1(x_1)x_2 + x_2[-h_1(x_1) - h_2(x_2)] = -x_2h_2(x_2) \leq 0$$

$$\dot{V}(x) = 0 \Rightarrow x_2h_2(x_2) = 0 \Rightarrow x_2 = 0$$

$$S = \{x \in D \mid x_2 = 0\}$$

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -h_1(x_1) - h_2(x_2)$$

$$x_2(t) \equiv 0 \Rightarrow \dot{x}_2(t) \equiv 0 \Rightarrow h_1(x_1(t)) \equiv 0 \Rightarrow x_1(t) \equiv 0$$

The only solution that can stay identically in S is $x(t) \equiv 0$

Thus, the origin is asymptotically stable

Suppose $a = \infty$ and $\int_0^y h_1(z) dz \rightarrow \infty$ as $|y| \rightarrow \infty$

Then, $D = \mathbb{R}^2$ and $V(x) = \int_0^{x_1} h_1(y) dy + \frac{1}{2}x_2^2$ is radially unbounded. $S = \{x \in \mathbb{R}^2 \mid x_2 = 0\}$ and the only solution that can stay identically in S is $x(t) \equiv 0$

The origin is globally asymptotically stable

Exponential Stability

The origin of $\dot{x} = f(x)$ is exponentially stable if and only if the linearization of $f(x)$ at the origin is Hurwitz

Theorem 3.6

Let $f(x)$ be a locally Lipschitz function defined over a domain $D \subset \mathbb{R}^n$; $0 \in D$. Let $V(x)$ be a continuously differentiable function such that

$$k_1\|x\|^a \leq V(x) \leq k_2\|x\|^a, \quad \dot{V}(x) \leq -k_3\|x\|^a$$

for all $x \in D$, where k_1 , k_2 , k_3 , and a are positive constants. Then, the origin is an exponentially stable equilibrium point of $\dot{x} = f(x)$. If the assumptions hold globally, the origin will be globally exponentially stable

Example 3.10

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -h(x_1) - x_2$$

$$c_1 y^2 \leq y h(y) \leq c_2 y^2, \quad \forall y, \quad c_1 > 0, \quad c_2 > 0$$

$$V(x) = \frac{1}{2} x^T \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} x + 2 \int_0^{x_1} h(y) dy$$

$$c_1 x_1^2 \leq 2 \int_0^{x_1} h(y) dy \leq c_2 x_1^2$$

$$\begin{aligned} \dot{V} &= [x_1 + x_2 + 2h(x_1)]x_2 + [x_1 + 2x_2][-h(x_1) - x_2] \\ &= -x_1 h(x_1) - x_2^2 \leq -c_1 x_1^2 - x_2^2 \end{aligned}$$

The origin is globally exponentially stable

Quadratic Forms

$$V(x) = x^T P x = \sum_{i=1}^n \sum_{j=1}^n p_{ij} x_i x_j, \quad P = P^T$$

$$\lambda_{\min}(P) \|x\|^2 \leq x^T P x \leq \lambda_{\max}(P) \|x\|^2$$

$P \geq 0$ (Positive semidefinite) if and only if $\lambda_i(P) \geq 0 \ \forall i$

$P > 0$ (Positive definite) if and only if $\lambda_i(P) > 0 \ \forall i$

$P > 0$ if and only if all the leading principal minors of P are positive

Linear Systems

$$\dot{x} = Ax$$

$$V(x) = x^T P x, \quad P = P^T > 0$$

$$\dot{V}(x) = x^T P \dot{x} + \dot{x}^T P x = x^T (PA + A^T P)x \stackrel{\text{def}}{=} -x^T Q x$$

If $Q > 0$, then A is Hurwitz

Or choose $Q > 0$ and solve the Lyapunov equation

$$PA + A^T P = -Q$$

If $P > 0$, then A is Hurwitz

MATLAB: $P = \text{lyap}(A', Q)$

Theorem 3.7

A matrix A is Hurwitz if and only if for every $Q = Q^T > 0$ there is $P = P^T > 0$ that satisfies the Lyapunov equation

$$PA + A^T P = -Q$$

Moreover, if A is Hurwitz, then P is the unique solution

Linearization

$$\dot{x} = f(x) = [A + G(x)]x$$

$$G(x) \rightarrow 0 \quad \text{as} \quad x \rightarrow 0$$

Suppose A is Hurwitz. Choose $Q = Q^T > 0$ and solve $PA + A^T P = -Q$ for P . Use $V(x) = x^T P x$ as a Lyapunov function candidate for $\dot{x} = f(x)$

$$\begin{aligned}\dot{V}(x) &= x^T P f(x) + f^T(x) P x \\ &= x^T P [A + G(x)]x + x^T [A^T + G^T(x)] P x \\ &= x^T (PA + A^T P)x + 2x^T P G(x)x \\ &= -x^T Q x + 2x^T P G(x)x\end{aligned}$$

$$\dot{V}(x) \leq -x^T Q x + 2\|P G(x)\| \|x\|^2$$

Given any positive constant $k < 1$, we can find $r > 0$ such that

$$2\|PG(x)\| < k\lambda_{\min}(Q), \quad \forall \|x\| < r$$

$$x^T Q x \geq \lambda_{\min}(Q)\|x\|^2 \iff -x^T Q x \leq -\lambda_{\min}(Q)\|x\|^2$$

$$\dot{V}(x) \leq -(1 - k)\lambda_{\min}(Q)\|x\|^2, \quad \forall \|x\| < r$$

$V(x) = x^T P x$ is a Lyapunov function for $\dot{x} = f(x)$

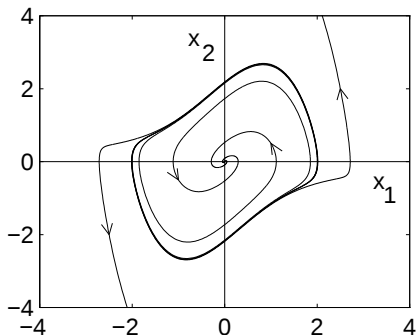
Region of Attraction

Lemma 3.2

The region of attraction of an asymptotically stable equilibrium point is an open, connected, invariant set, and its boundary is formed by trajectories

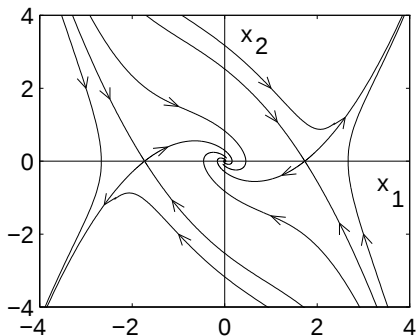
Example 3.11

$$\dot{x}_1 = -x_2, \quad \dot{x}_2 = x_1 + (x_1^2 - 1)x_2$$



Example 3.12

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + \frac{1}{3}x_1^3 - x_2$$



By Theorem 3.5, if D is a domain that contains the origin such that $\dot{V}(x) \leq 0$ in D , then the region of attraction can be estimated by a compact positively invariant set $\Gamma \in D$ if

- $\dot{V}(x) < 0$ for all $x \in \Gamma$, $x \neq 0$, or
- No solution can stay identically in $\{x \in D \mid \dot{V}(x) = 0\}$ other than the zero solution.

The simplest such estimate is the set $\Omega_c = \{V(x) \leq c\}$ when Ω_c is bounded and contained in D

$$V(x) = x^T P x, \quad P = P^T > 0, \quad \Omega_c = \{V(x) \leq c\}$$

If $D = \{\|x\| < r\}$, then $\Omega_c \subset D$ if

$$c < \min_{\|x\|=r} x^T P x = \lambda_{\min}(P) r^2$$

If $D = \{|b^T x| < r\}$, where $b \in R^n$, then

$$\min_{|b^T x|=r} x^T P x = \frac{r^2}{b^T P^{-1} b}$$

Therefore, $\Omega_c \subset D = \{|b_i^T x| < r_i, i = 1, \dots, p\}$, if

$$c < \min_{1 \leq i \leq p} \frac{r_i^2}{b_i^T P^{-1} b_i}$$

Example 3.14

$$\dot{x}_1 = -x_2, \quad \dot{x}_2 = x_1 + (x_1^2 - 1)x_2$$

$$A = \left. \frac{\partial f}{\partial x} \right|_{x=0} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

has eigenvalues $(-1 \pm j\sqrt{3})/2$. Hence the origin is asymptotically stable

$$\text{Take } Q = I, \quad PA + A^T P = -I \Rightarrow P = \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 1 \end{bmatrix}$$

$$\lambda_{\min}(P) = 0.691$$

$$V(x) = 1.5x_1^2 - x_1x_2 + x_2^2$$

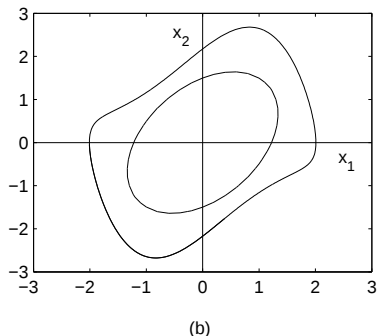
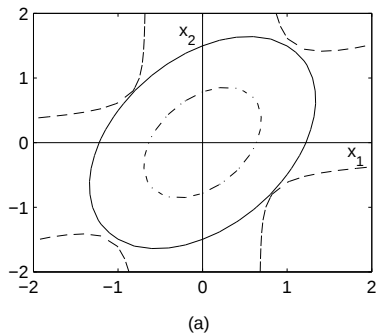
$$\dot{V}(x) = -(x_1^2 + x_2^2) - x_1^2x_2(x_1 - 2x_2)$$

$$|x_1| \leq \|x\|, \quad |x_1x_2| \leq \frac{1}{2}\|x\|^2, \quad |x_1 - 2x_2| \leq \sqrt{5}\|x\|$$

$$\dot{V}(x) \leq -\|x\|^2 + \frac{\sqrt{5}}{2}\|x\|^4 < 0 \quad \text{for } 0 < \|x\|^2 < \frac{2}{\sqrt{5}} \stackrel{\text{def}}{=} r^2$$

$$\text{Take } c < \lambda_{\min}(P)r^2 = 0.691 \times \frac{2}{\sqrt{5}} = 0.618$$

$\{V(x) \leq c\}$ is an estimate of the region of attraction



(a) Contours of $\dot{V}(x) = 0$ (dashed)

$V(x) = 0.618$ (dash-dot), $V(x) = 2.25$ (solid)

(b) comparison of the region of attraction with its estimate

Remark 3.1

If $\Omega_1, \Omega_2, \dots, \Omega_m$ are positively invariant subsets of the region of attraction, then their union $\cup_{i=1}^m \Omega_i$ is also a positively invariant subset of the region of attraction. Therefore, if we have multiple Lyapunov functions for the same system and each function is used to estimate the region of attraction, we can enlarge the estimate by taking the union of all the estimates

Remark 3.2

we can work with any compact set $\Gamma \subset D$ provided we can show that Γ is positively invariant. This typically requires investigating the vector field at the boundary of Γ to ensure that trajectories starting in Γ cannot leave it

Example 3.15 (Read)

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -4(x_1 + x_2) - h(x_1 + x_2)$$

$$h(0) = 0; \quad uh(u) \geq 0, \quad \forall |u| \leq 1$$

$$V(x) = x^T P x = x^T \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} x = 2x_1^2 + 2x_1x_2 + x_2^2$$

$$\begin{aligned} \dot{V}(x) &= (4x_1 + 2x_2)\dot{x}_1 + 2(x_1 + x_2)\dot{x}_2 \\ &= -2x_1^2 - 6(x_1 + x_2)^2 - 2(x_1 + x_2)h(x_1 + x_2) \\ &\leq -2x_1^2 - 6(x_1 + x_2)^2, \quad \forall |x_1 + x_2| \leq 1 \\ &= -x^T \begin{bmatrix} 8 & 6 \\ 6 & 6 \end{bmatrix} x \end{aligned}$$

$$V(x) = x^T P x = x^T \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} x$$

$\dot{V}(x)$ is negative definite in $\{|x_1 + x_2| \leq 1\}$

$$b^T = [1 \ 1], \quad c = \min_{|x_1+x_2|=1} x^T P x = \frac{1}{b^T P^{-1} b} = 1$$

The region of attraction is estimated by $\{V(x) \leq 1\}$

$$\sigma = x_1 + x_2$$

$$\frac{d}{dt}\sigma^2 = 2\sigma x_2 - 8\sigma^2 - 2\sigma h(\sigma) \leq 2\sigma x_2 - 8\sigma^2, \quad \forall |\sigma| \leq 1$$

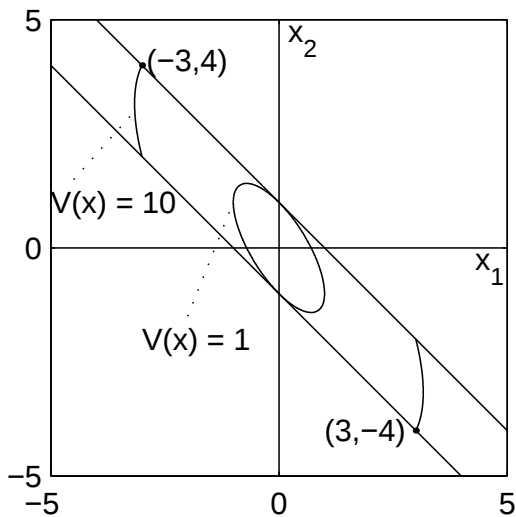
$$\text{On } \sigma = 1, \quad \frac{d}{dt}\sigma^2 \leq 2x_2 - 8 \leq 0, \quad \forall x_2 \leq 4$$

$$\text{On } \sigma = -1, \quad \frac{d}{dt}\sigma^2 \leq -2x_2 - 8 \leq 0, \quad \forall x_2 \geq -4$$

$$c_1 = V(x)|_{x_1=-3, x_2=4} = 10, \quad c_2 = V(x)|_{x_1=3, x_2=-4} = 10$$

$$\Gamma = \{V(x) \leq 10 \text{ and } |x_1 + x_2| \leq 1\}$$

is a subset of the region of attraction



Converse Lyapunov Theorems

Theorem 3.8 (Exponential Stability)

Let $x = 0$ be an exponentially stable equilibrium point for the system $\dot{x} = f(x)$, where f is continuously differentiable on $D = \{\|x\| < r\}$. Let k , λ , and r_0 be positive constants with $r_0 < r/k$ such that

$$\|x(t)\| \leq k\|x(0)\|e^{-\lambda t}, \quad \forall x(0) \in D_0, \quad \forall t \geq 0$$

where $D_0 = \{\|x\| < r_0\}$. Then, there is a continuously differentiable function $V(x)$ that satisfies the inequalities

$$c_1 \|x\|^2 \leq V(x) \leq c_2 \|x\|^2$$

$$\frac{\partial V}{\partial x} f(x) \leq -c_3 \|x\|^2$$

$$\left\| \frac{\partial V}{\partial x} \right\| \leq c_4 \|x\|$$

for all $x \in D_0$, with positive constants c_1 , c_2 , c_3 , and c_4 .
Moreover, if f is continuously differentiable for all x , globally Lipschitz, and the origin is globally exponentially stable, then $V(x)$ is defined and satisfies the aforementioned inequalities for all $x \in \mathbb{R}^n$.

Example 3.16 (Read)

Consider the system $\dot{x} = f(x)$ where f is continuously differentiable in the neighborhood of the origin and $f(0) = 0$. Show that the origin is exponentially stable only if $A = [\partial f / \partial x](0)$ is Hurwitz

$$f(x) = Ax + G(x)x, \quad G(x) \rightarrow 0 \text{ as } x \rightarrow 0$$

Given any $L > 0$, there is $r_1 > 0$ such that

$$\|G(x)\| \leq L, \quad \forall \|x\| < r_1$$

Because the origin of $\dot{x} = f(x)$ is exponentially stable, let $V(x)$ be the function provided by the converse Lyapunov theorem over the domain $\{\|x\| < r_0\}$. Use $V(x)$ as a Lyapunov function candidate for $\dot{x} = Ax$

$$\begin{aligned}
 \frac{\partial V}{\partial x} Ax &= \frac{\partial V}{\partial x} f(x) - \frac{\partial V}{\partial x} G(x)x \\
 &\leq -c_3 \|x\|^2 + c_4 L \|x\|^2 \\
 &= -(c_3 - c_4 L) \|x\|^2
 \end{aligned}$$

Take $L < c_3/c_4$, $\gamma \stackrel{\text{def}}{=} (c_3 - c_4 L) > 0 \Rightarrow$

$$\frac{\partial V}{\partial x} Ax \leq -\gamma \|x\|^2, \quad \forall \|x\| < \min\{r_0, r_1\}$$

The origin of $\dot{x} = Ax$ is exponentially stable

Theorem 3.9 (Asymptotic Stability)

Let $x = 0$ be an asymptotically stable equilibrium point for $\dot{x} = f(x)$, where f is locally Lipschitz on a domain $D \subset \mathbb{R}^n$ that contains the origin. Let $R_A \subset D$ be the region of attraction of $x = 0$. Then, there is a smooth, positive definite function $V(x)$ and a continuous, positive definite function $W(x)$, both defined for all $x \in R_A$, such that

$$V(x) \rightarrow \infty \text{ as } x \rightarrow \partial R_A$$

$$\frac{\partial V}{\partial x} f(x) \leq -W(x), \quad \forall x \in R_A$$

and for any $c > 0$, $\{V(x) \leq c\}$ is a compact subset of R_A
When $R_A = \mathbb{R}^n$, $V(x)$ is radially unbounded