Nonlinear Control

Lecture # 8

Time Varying

and

Perturbed Systems

Time-varying Systems

$$\dot{x} = f(t, x)$$

$f(t, x)$ is piecewise continuous in t and locally Lipschitz in x for all $t \geq 0$ and all $x \in D$, $(0 \in D)$. The origin is an equilibrium point at $t = 0$ if

$$f(t, 0) = 0, \quad \forall \; t \geq 0$$

While the solution of the time-invariant system

$$\dot{x} = f(x), \qquad x(t_0) = x_0$$

depends only on $(t - t_0)$, the solution of

$$\dot{x} = f(t, x), \qquad x(t_0) = x_0$$

may depend on both t and t_0

Comparison Functions

- A scalar continuous function $\alpha(r)$, defined for $r \in [0, a)$, belongs to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It belongs to class $\mathcal{K}_\infty$ if it is defined for all $r \geq 0$ and $\alpha(r) \to \infty$ as $r \to \infty$
- A scalar continuous function $\beta(r, s)$, defined for $r \in [0, a)$ and $s \in [0, \infty)$, belongs to class $\mathcal{KL}$ if, for each fixed s, the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r and, for each fixed r, the mapping $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$

Example 4.1

- $\alpha(r) = \tan^{-1}(r)$ is strictly increasing since $\alpha'(r) = 1/(1 + r^2) > 0$. It belongs to class $\mathcal{K}$, but not to class $\mathcal{K}_\infty$ since $\lim_{r\to\infty} \alpha(r) = \pi/2 < \infty$
- $\alpha(r) = r^c$, $c > 0$, is strictly increasing since $\alpha'(r) = cr^{c-1} > 0$. Moreover, $\lim_{r\to\infty} \alpha(r) = \infty$; thus, it belongs to class $\mathcal{K}_\infty$
- $\alpha(r) = \min\{r, r^2\}$ is continuous, strictly increasing, and $\lim_{r\to\infty} \alpha(r) = \infty$. Hence, it belongs to class $\mathcal{K}_\infty$. It is not continuously differentiable at $r = 1$. Continuous differentiability is not required for a class $\mathcal{K}$ function

- $\beta(r,s) = r/(ksr+1)$, for any positive constant k, is strictly increasing in r since

$$\frac{\partial \beta}{\partial r} = \frac{1}{(ksr+1)^2} > 0$$

and strictly decreasing in s since

$$\frac{\partial \beta}{\partial s} = \frac{-kr^2}{(ksr+1)^2} < 0$$

$\beta(r,s) \to 0$ as $s \to \infty$. It belongs to class $\mathcal{KL}$

- $\beta(r,s) = r^c e^{-as}$, with positive constants a and c, belongs to class $\mathcal{KL}$

Lemma 4.1

Let α_1 and α_2 be class $\mathcal{K}$ functions on $[0, a_1)$ and $[0, a_2)$, respectively, with $a_1 \geq \lim_{r \to a_2} \alpha_2(r)$, and β be a class $\mathcal{KL}$ function defined on $[0, \lim_{r \to a_2} \alpha_2(r)) \times [0, \infty)$ with $a_1 \geq \lim_{r \to a_2} \beta(\alpha_2(r), 0)$. Let α_3 and α_4 be class $\mathcal{K}_\infty$ functions. Denote the inverse of α_i by α_i^{-1}. Then,

- α_1^{-1} is defined on $[0, \lim_{r \to a_1} \alpha_1(r))$ and belongs to class $\mathcal{K}$
- α_3^{-1} is defined on $[0, \infty)$ and belongs to class $\mathcal{K}_\infty$
- $\alpha_1 \circ \alpha_2$ is defined on $[0, a_2)$ and belongs to class $\mathcal{K}$
- $\alpha_3 \circ \alpha_4$ is defined on $[0, \infty)$ and belongs to class $\mathcal{K}_\infty$
- $\sigma(r, s) = \alpha_1(\beta(\alpha_2(r), s))$ is defined on $[0, a_2) \times [0, \infty)$ and belongs to class $\mathcal{KL}$

Lemma 4.2

Let $V : D \to R$ be a continuous positive definite function defined on a domain $D \subset R^n$ that contains the origin. Let $B_r \subset D$ for some $r > 0$. Then, there exist class $\mathcal{K}$ functions α_1 and α_2, defined on $[0, r]$, such that

$$\alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|)$$

for all $x \in B_r$. If $D = R^n$ and $V(x)$ is radially unbounded, then there exist class $\mathcal{K}_\infty$ functions α_1 and α_2 such that the foregoing inequality holds for all $x \in R^n$

Definition 4.2

The equilibrium point $x = 0$ of $\dot{x} = f(t, x)$ is

- uniformly stable if there exist a class $\mathcal{K}$ function α and a positive constant c, independent of t_0, such that

$$\|x(t)\| \leq \alpha(\|x(t_0)\|), \ \forall\ t \geq t_0 \geq 0, \ \forall\ \|x(t_0)\| < c$$

- uniformly asymptotically stable if there exist a class $\mathcal{KL}$ function β and a positive constant c, independent of t_0, such that

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \ \forall\ t \geq t_0 \geq 0, \ \forall\ \|x(t_0)\| < c$$

- globally uniformly asymptotically stable if the foregoing inequality is satisfied for any initial state $x(t_0)$

- exponentially stable if there exist positive constants c, k, and λ such that

$$\|x(t)\| \le k\|x(t_0)\|e^{-\lambda(t-t_0)}, \ \forall \ \|x(t_0)\| < c$$

- globally exponentially stable if the foregoing inequality is satisfied for any initial state $x(t_0)$

Theorem 4.1

Let the origin $x = 0$ be an equilibrium point of $\dot{x} = f(t, x)$ and $D \subset R^n$ be a domain containing $x = 0$. Suppose $f(t, x)$ is piecewise continuous in t and locally Lipschitz in x for all $t \geq 0$ and $x \in D$. Let $V(t, x)$ be a continuously differentiable function such that

$$W_1(x) \leq V(t, x) \leq W_2(x)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq 0$$

for all $t \geq 0$ and $x \in D$, where $W_1(x)$ and $W_2(x)$ are continuous positive definite functions on D. Then, the origin is uniformly stable

Theorem 4.2

Suppose the assumptions of the previous theorem are satisfied with

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -W_3(x)$$

for all $t \geq 0$ and $x \in D$, where $W_3(x)$ is a continuous positive definite function on D. Then, the origin is uniformly asymptotically stable. Moreover, if r and c are chosen such that $B_r = \{\|x\| \leq r\} \subset D$ and $c < \min_{\|x\|=r} W_1(x)$, then every trajectory starting in $\{W_2(x) \leq c\}$ satisfies

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \quad \forall\ t \geq t_0 \geq 0$$

for some class $\mathcal{KL}$ function β. Finally, if $D = R^n$ and $W_1(x)$ is radially unbounded, then the origin is globally uniformly asymptotically stable

Theorem 4.3

Suppose the assumptions of the previous theorem are satisfied with

$$k_1\|x\|^a \le V(t,x) \le k_2\|x\|^a$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \le -k_3\|x\|^a$$

for all $t \ge 0$ and $x \in D$, where k_1, k_2, k_3, and a are positive constants. Then, the origin is exponentially stable. If the assumptions hold globally, the origin will be globally exponentially stable

Terminology: A function $V(t,x)$ is said to be

- positive semidefinite if $V(t,x) \geq 0$
- positive definite if $V(t,x) \geq W_1(x)$ for some positive definite function $W_1(x)$
- radially unbounded if $V(t,x) \geq W_1(x)$ and $W_1(x)$ is radially unbounded
- decrescent if $V(t,x) \leq W_2(x)$
- negative definite (semidefinite) if $-V(t,x)$ is positive definite (semidefinite)

Example 4.2

$$\dot{x} = -[1 + g(t)]x^3, \quad g(t) \ge 0, \ \forall \ t \ge 0$$

$$V(x) = \tfrac{1}{2}x^2$$

$$\dot{V}(t, x) = -[1 + g(t)]x^4 \le -x^4, \quad \forall \ x \in R, \ \forall \ t \ge 0$$

The origin is globally uniformly asymptotically stable

Example 4.3

$$\dot{x}_1 = -x_1 - g(t)x_2, \qquad \dot{x}_2 = x_1 - x_2$$

$$0 \le g(t) \le k \ \text{ and } \ \dot{g}(t) \le g(t), \ \forall \ t \ge 0$$

$$V(t,x) = x_1^2 + [1+g(t)]x_2^2$$

$$x_1^2 + x_2^2 \le V(t,x) \le x_1^2 + (1+k)x_2^2, \quad \forall\ x \in R^2$$

$$\dot{V}(t,x) = -2x_1^2 + 2x_1x_2 - [2 + 2g(t) - \dot{g}(t)]x_2^2$$

$$2 + 2g(t) - \dot{g}(t) \ge 2 + 2g(t) - g(t) \ge 2$$

$$\dot{V}(t,x) \le -2x_1^2 + 2x_1x_2 - 2x_2^2 = -\ x^T \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} x$$

The origin is globally exponentially stable

Perturbed Systems

Nominal System:

$$\dot{x} = f(x), \qquad f(0) = 0$$

Perturbed System:

$$\dot{x} = f(x) + g(t, x), \qquad g(t, 0) = 0$$

Case 1: The origin of the nominal system is exponentially stable

$$c_1\|x\|^2 \leq V(x) \leq c_2\|x\|^2$$

$$\frac{\partial V}{\partial x} f(x) \leq -c_3\|x\|^2$$

$$\left\|\frac{\partial V}{\partial x}\right\| \leq c_4\|x\|$$

Use $V(x)$ as a Lyapunov function candidate for the perturbed system

$$\dot{V}(t,x) = \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(t,x)$$

Assume that

$$\|g(t,x)\| \leq \gamma \|x\|, \qquad \gamma \geq 0$$

$$\begin{aligned} \dot{V}(t,x) &\leq -c_3 \|x\|^2 + \left\| \frac{\partial V}{\partial x} \right\| \, \|g(t,x)\| \\ &\leq -c_3 \|x\|^2 + c_4 \gamma \|x\|^2 \end{aligned}$$

$$\gamma < \frac{c_3}{c_4}$$

$$\dot{V}(t,x) \le -(c_3 - \gamma c_4)\|x\|^2$$

The origin is an exponentially stable equilibrium point of the perturbed system

Example 4.4

$$\dot{x} = Ax + g(t,x); \quad A \text{ is Hurwitz}; \quad \|g(t,x)\| \le \gamma \|x\|$$

$$Q = Q^T > 0; \quad PA + A^T P = -Q; \quad V(x) = x^T P x$$

$$\lambda_{\min}(P)\|x\|^2 \le V(x) \le \lambda_{\max}(P)\|x\|^2$$

$$\frac{\partial V}{\partial x} Ax = -x^T Q x \le -\lambda_{\min}(Q)\|x\|^2$$

$$\left\| \frac{\partial V}{\partial x} g \right\| = \|2x^T P g\| \le 2\|P\| \|x\| \|g\| \le 2\|P\|\gamma\|x\|^2$$

$$\dot{V}(t,x) \le -\lambda_{\min}(Q)\|x\|^2 + 2\lambda_{\max}(P)\gamma\|x\|^2$$

The origin is globally exponentially stable if $\gamma < \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}$

Example 4.5

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -4x_1 - 2x_2 + \beta x_2^3, \qquad \beta \geq 0\end{aligned}$$

$$\dot{x} = Ax + g(x)$$

$$A = \begin{bmatrix} 0 & 1 \\ -4 & -2 \end{bmatrix}, \qquad g(x) = \begin{bmatrix} 0 \\ \beta x_2^3 \end{bmatrix}$$

The eigenvalues of A are $-1 \pm j\sqrt{3}$

$$PA + A^T P = -I \quad \Rightarrow \quad P = \begin{bmatrix} \frac{3}{2} & \frac{1}{8} \\ \frac{1}{8} & \frac{5}{16} \end{bmatrix}$$

$$V(x) = x^T P x, \qquad \frac{\partial V}{\partial x} A x = -x^T x$$

$$c_3 = 1, \quad c_4 = 2\,\|P\| = 2\lambda_{\max}(P) = 2 \times 1.513 = 3.026$$

$$\|g(x)\| = \beta |x_2|^3$$

$g(x)$ satisfies the bound $\|g(x)\| \leq \gamma \|x\|$ over compact sets of x. Consider the compact set

$$\Omega_c = \{V(x) \leq c\} = \{x^T P x \leq c\}, \quad c > 0$$

$$\begin{aligned} k_2 &= \max_{x^T P x \leq c} |x_2| = \max_{x^T P x \leq c} \left| \begin{bmatrix} 0 & 1 \end{bmatrix} x \right| \\ &= \sqrt{c}\, \left\| \begin{bmatrix} 0 & 1 \end{bmatrix} P^{-1/2} \right\| = 1.8194\sqrt{c} \end{aligned}$$

$$k_2 = \max_{x^T P x \leq c} |[0 \;\; 1]x| = 1.8194\sqrt{c}$$

$$\|g(x)\| \leq \beta \; c \; (1.8194)^2 \|x\|, \quad \forall \; x \in \Omega_c$$

$$\|g(x)\| \leq \gamma \|x\|, \quad \forall \; x \in \Omega_c, \quad \gamma = \beta \; c \; (1.8194)^2$$

$$\gamma < \frac{c_3}{c_4} \;\; \Leftrightarrow \;\; \beta < \frac{1}{3.026 \times (1.8194)^2 c} \approx \frac{0.1}{c}$$

$$\beta < 0.1/c \;\; \Rightarrow \;\; \dot{V}(x) \leq -(1 - 10\beta c)\|x\|^2$$

Hence, the origin is exponentially stable and Ω_c is an estimate of the region of attraction

Alternative Bound on β:

$$\begin{aligned}\dot{V}(x) &= -\|x\|^2 + 2x^T P g(x) \;\leq\; -\|x\|^2 + \tfrac{1}{8}\beta x_2^3\left([2 \quad 5]x\right) \\ &\leq -\|x\|^2 + \tfrac{\sqrt{29}}{8}\beta x_2^2 \|x\|^2\end{aligned}$$

Over Ω_c, $x_2^2 \leq (1.8194)^2 c$

$$\begin{aligned}\dot{V}(x) &\leq -\left(1 - \tfrac{\sqrt{29}}{8}\beta (1.8194)^2 c\right)\|x\|^2 \\ &= -\left(1 - \frac{\beta c}{0.448}\right)\|x\|^2\end{aligned}$$

If $\beta < 0.448/c$, the origin will be exponentially stable and Ω_c will be an estimate of the region of attraction

Remark

The inequality $\beta < 0.448/c$ shows a tradeoff between the estimate of the region of attraction and the estimate of the upper bound on β

Case 2: The origin of the nominal system is asymptotically stable

$$\dot{V}(t,x) = \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(t,x) \le -W_3(x) + \left\| \frac{\partial V}{\partial x} g(t,x) \right\|$$

Under what condition will the following inequality hold?

$$\left\| \frac{\partial V}{\partial x} g(t,x) \right\| < W_3(x)$$

Special Case: Quadratic-Type Lyapunov function

$$\frac{\partial V}{\partial x} f(x) \le -c_3 \phi^2(x), \quad \left\| \frac{\partial V}{\partial x} \right\| \le c_4 \phi(x)$$

$$\dot{V}(t,x) \le -c_3\phi^2(x) + c_4\phi(x)\|g(t,x)\|$$

If $\|g(t,x)\| \le \gamma\phi(x), \quad \text{with } \gamma < \dfrac{c_3}{c_4}$

$$\dot{V}(t,x) \le -(c_3 - c_4\gamma)\phi^2(x)$$

Example 4.6

$$\dot{x} = -x^3 + g(t,x)$$

$V(x) = x^4$ is a quadratic-type Lyapunov function for $\dot{x} = -x^3$

$$\frac{\partial V}{\partial x}(-x^3) = -4x^6, \quad \left|\frac{\partial V}{\partial x}\right| = 4|x|^3$$

$$\phi(x) = |x|^3, \quad c_3 = 4, \quad c_4 = 4$$

Suppose $|g(t,x)| \le \gamma |x|^3, \ \forall\, x,$ with $\gamma < 1$

$$\dot{V}(t,x) \le -4(1-\gamma)\phi^2(x)$$

Hence, the origin is a globally uniformly asymptotically stable

Remark

A nominal system with asymptotically, but not exponentially, stable origin is not robust to smooth perturbations with arbitrarily small linear growth bounds

Example 4.7

$$\dot{x} = -x^3 + \gamma x$$

The origin is unstable for any $\gamma > 0$