Nonlinear Control

Lecture # 5
Time Varying
and
Perturbed Systems

Boundedness and Ultimate Boundedness

Definition 4.3

The solutions of $\dot{x} = f(t, x)$ are

- uniformly bounded if there exists $c > 0$, independent of t_0, and for every $a \in (0, c)$, there is $\beta > 0$, dependent on a but independent of t_0, such that

$$\|x(t_0)\| \leq a \ \Rightarrow \ \|x(t)\| \leq \beta, \ \ \forall \ t \geq t_0$$

- uniformly ultimately bounded with ultimate bound b if there exists a positive constant c, independent of t_0, and for every $a \in (0, c)$, there is $T \geq 0$, dependent on a and b but independent of t_0, such that

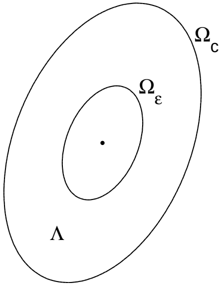
$$\|x(t_0)\| \leq a \ \Rightarrow \ \|x(t)\| \leq b, \ \ \forall \ t \geq t_0 + T$$

- Add "Globally" if a can be arbitrarily large
- Drop "uniformly" if $\dot{x} = f(x)$

Lyapunov Analysis: Let $V(x)$ be a cont. diff. positive definite function and suppose the sets

$$\Omega_c = \{V(x) \le c\},\ \Omega_\varepsilon = \{V(x) \le \varepsilon\},\ \Lambda = \{\varepsilon \le V(x) \le c\}$$

are compact for some $c > \varepsilon > 0$



Suppose

$$\dot{V}(t,x) = \frac{\partial V}{\partial x} f(t,x) \le -W_3(x), \quad \forall\ x \in \Lambda,\ \forall\ t \ge 0$$

$W_3(x)$ is continuous and positive definite

Ω_c and Ω_ε are positively invariant

$$k = \min_{x\in\Lambda} W_3(x) > 0$$

$$\dot{V}(t,x) \le -k, \quad \forall\ x \in \Lambda,\ \forall\ t \ge t_0 \ge 0$$

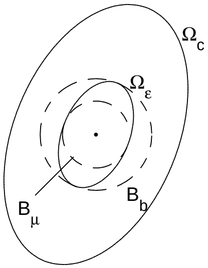
$$V(x(t)) \le V(x(t_0)) - k(t-t_0) \le c - k(t-t_0)$$

$x(t)$ enters the set Ω_ε within the interval $[t_0, t_0 + (c-\varepsilon)/k]$

Suppose

$$\dot{V}(t,x) \le -W_3(x), \quad \forall\, x \in D \text{ with } \|x\| \ge \mu,\ \forall\, t \ge 0$$

Choose c and ε such that $\Lambda \subset D \cap \{\|x\| \ge \mu\}$



Let α_1 and α_2 be class $\mathcal{K}$ functions such that

$$\alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|)$$

$$V(x) \le c \;\Rightarrow\; \alpha_1(\|x\|) \le c \;\Leftrightarrow\; \|x\| \le \alpha_1^{-1}(c)$$

$$\text{If } B_r \subset D, \quad c = \alpha_1(r) \quad \Rightarrow \quad \Omega_c \subset B_r \subset D$$

$$\|x\| \le \mu \;\Rightarrow\; V(x) \le \alpha_2(\mu)$$

$$\varepsilon = \alpha_2(\mu) \quad \Rightarrow \quad B_\mu \subset \Omega_\varepsilon$$

What is the ultimate bound?

$$V(x) \le \varepsilon \Rightarrow \alpha_1(\|x\|) \le \varepsilon \Leftrightarrow \|x\| \le \alpha_1^{-1}(\varepsilon) = \alpha_1^{-1}(\alpha_2(\mu))$$

Theorem 4.4

Suppose $B_\mu \subset D \subset R^n$ and

$$\alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|)$$

$$\frac{\partial V}{\partial x} f(t,x) \le -W_3(x), \quad \forall\ x \in D \text{ with } \|x\| \ge \mu,\ \forall\ t \ge 0$$

where α_1 and α_2 are class $\mathcal{K}$ functions and $W_3(x)$ is a continuous positive definite function. Choose $c > 0$ such that $\Omega_c = \{V(x) \le c\}$ is compact and contained in D and suppose $\mu < \alpha_2^{-1}(c)$. Then, Ω_c is positively invariant and there exists a class $\mathcal{KL}$ function β such that for every $x(t_0) \in \Omega_c$,

$$\|x(t)\| \le \max\left\{\beta(\|x(t_0)\|, t-t_0), \alpha_1^{-1}(\alpha_2(\mu))\right\}, \quad \forall\ t\ \ge t_0$$

If $D = R^n$ and $\alpha_1 \in \mathcal{K}_\infty$, the inequality holds $\forall x(t_0)$, $\forall \mu$

Remarks

- The ultimate bound is independent of the initial state
- The ultimate bound is a class $\mathcal{K}$ function of μ; hence, the smaller the value of μ, the smaller the ultimate bound. As $\mu \to 0$, the ultimate bound approaches zero

Example 4.8

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -(1 + x_1^2)x_1 - x_2 + M\cos\omega t, \quad M \geq 0$$

With $M = 0, \quad \dot{x}_2 = -(1 + x_1^2)x_1 - x_2 = -h(x_1) - x_2$

$$V(x) = x^T \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} x + 2\int_0^{x_1} (y + y^3)\ dy \quad \text{(Example 3.7)}$$

$$V(x) = x^T \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} x + \tfrac{1}{2}x_1^4 \stackrel{\text{def}}{=} x^T P x + \tfrac{1}{2}x_1^4$$

$$\lambda_{min}(P)\|x\|^2 \leq V(x) \leq \lambda_{max}(P)\|x\|^2 + \tfrac{1}{2}\|x\|^4$$

$$\alpha_1(r) = \lambda_{min}(P)r^2, \qquad \alpha_2(r) = \lambda_{max}(P)r^2 + \tfrac{1}{2}r^4$$

$$\begin{aligned}
\dot{V} &= -x_1^2 - x_1^4 - x_2^2 + (x_1 + 2x_2)M\cos\omega t \\
&\leq -\|x\|^2 - x_1^4 + M\sqrt{5}\|x\| \\
&= -(1-\theta)\|x\|^2 - x_1^4 - \theta\|x\|^2 + M\sqrt{5}\|x\| \\
&\qquad\qquad\qquad\qquad\qquad\qquad (0 < \theta < 1) \\
&\leq -(1-\theta)\|x\|^2 - x_1^4, \;\; \forall\ \|x\| \geq M\sqrt{5}/\theta \stackrel{\text{def}}{=} \mu
\end{aligned}$$

The solutions are GUUB by

$$b = \alpha_1^{-1}(\alpha_2(\mu)) = \sqrt{\frac{\lambda_{max}(P)\mu^2 + \mu^4/2}{\lambda_{min}(P)}}$$

Theorem 4.5

Suppose

$$c_1\|x\|^2 \le V(x) \le c_2\|x\|^2$$

$$\frac{\partial V}{\partial x} f(t,x) \le -c_3\|x\|^2, \ \ \forall\ x \in D \text{ with } \|x\| \ge \mu, \ \forall\ t \ge 0$$

for some positive constants c_1 to c_3, and $\mu < \sqrt{c/c_2}$. Then, $\Omega_c = \{V(x) \le c\}$ is positively invariant and $\forall\ x(t_0) \in \Omega_c$

$$V(x(t)) \le \max\left\{V(x(t_0))e^{-(c_3/c_2)(t-t_0)},\ c_2\mu^2\right\}, \quad \forall\ t \ge t_0$$

$$\|x(t)\| \le \sqrt{c_2/c_1}\max\left\{\|x(t_0)\|e^{-(c_3/c_2)(t-t_0)/2},\ \mu\right\}, \ \forall\ t \ge t_0$$

If $D = R^n$, the inequalities hold $\forall x(t_0)$, $\forall \mu$

Example 4.9

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -h(x_1) - x_2 + u(t), \quad h(x_1) = x_1 - \tfrac{1}{3}x_1^3$$

$|u(t)| \leq d$

$$V(x) = \tfrac{1}{2}x^T \begin{bmatrix} k & k \\ k & 1 \end{bmatrix} x + \int_0^{x_1} h(y)\ dy, \quad 0 < k < 1$$

$$\tfrac{2}{3}x_1^2 \leq x_1 h(x_1) \leq x_1^2,\ \tfrac{5}{12}x_1^2 \leq \int_0^{x_1} h(y)\ dy \leq \tfrac{1}{2}x_1^2,\ \forall\ |x_1| \leq 1$$

$$\lambda_{\min}(P_1)\|x\|^2 \leq x^T P_1 x \leq V(x) \leq x^T P_2 x \leq \lambda_{\max}(P_2)\|x\|^2$$

$$P_1 = \tfrac{1}{2}\begin{bmatrix} k + \frac{5}{6} & k \\ k & 1 \end{bmatrix}, \quad P_2 = \tfrac{1}{2}\begin{bmatrix} k+1 & k \\ k & 1 \end{bmatrix}$$

$$\begin{aligned}\dot{V} &= -kx_1 h(x_1) - (1-k)x_2^2 + (kx_1 + x_2)u(t) \\ &\le -\tfrac{2}{3}kx_1^2 - (1-k)x_2^2 + |kx_1 + x_2|\ d\end{aligned}$$

$$k = \tfrac{3}{5} \quad \Rightarrow c_1 = \lambda_{\min}(P_1) = 0.2894,\ c_2 = \lambda_{\max}(P_2) = 0.9854$$

$$\begin{aligned}\dot{V} &\le -\tfrac{0.1\times 2}{5}\|x\|^2 - \tfrac{0.9\times 2}{5}\|x\|^2 + \sqrt{1 + \left(\tfrac{3}{5}\right)^2}\ \|x\|\ d \\ &\le \tfrac{0.1\times 2}{5}\|x\|^2, \quad \forall\ \|x\| \ge 3.2394\ d \stackrel{\text{def}}{=} \mu\end{aligned}$$

$$c = \min_{|x_1|=1} V(x) = 0.5367 \ \ \Rightarrow \ \ \Omega_c = \{V(x) \le c\} \subset \{|x_1| \le 1\}$$

For $\mu < \sqrt{c/c_2}$ we need $d < 0.2278$. Theorem 4.5 holds and $b = \mu\sqrt{c_2/c_1} = 5.9775\ d$

Perturbed Systems: Nonvanishing Perturbation

Nominal System:

$$\dot{x} = f(x), \qquad f(0) = 0$$

Perturbed System:

$$\dot{x} = f(x) + g(t, x), \qquad g(t, 0) \neq 0$$

Case 1:(Lemma 4.3) The origin of $\dot{x} = f(x)$ is exponentially stable

Case 2:(Lemma 4.4) The origin of $\dot{x} = f(x)$ is asymptotically stable

Lemma 4.3

Suppose that $\forall\ x \in B_r$, $\forall\ t \geq 0$

$$c_1\|x\|^2 \leq V(x) \leq c_2\|x\|^2$$

$$\frac{\partial V}{\partial x}f(x) \leq -c_3\|x\|^2, \quad \left\|\frac{\partial V}{\partial x}\right\| \leq c_4\|x\|$$

$$\|g(t,x)\| \leq \delta < \frac{c_3}{c_4}\sqrt{\frac{c_1}{c_2}}\theta r, \quad 0 < \theta < 1$$

Then, for all $x(t_0) \in \{V(x) \leq c_1 r^2\}$

$$\|x(t)\| \leq \max\left\{k\ \exp[-\gamma(t-t_0)]\|x(t_0)\|,\ b\right\}, \quad \forall\ t \geq t_0$$

$$k = \sqrt{\frac{c_2}{c_1}}, \quad \gamma = \frac{(1-\theta)c_3}{2c_2}, \quad b = \frac{\delta c_4}{\theta c_3}\sqrt{\frac{c_2}{c_1}}$$

Proof

Apply Theorem 4.5

$$\begin{aligned}\dot{V}(t,x) &= \tfrac{\partial V}{\partial x} f(x) + \tfrac{\partial V}{\partial x} g(t,x)\\ &\le -c_3\|x\|^2 + \left\|\tfrac{\partial V}{\partial x}\right\| \ \|g(t,x)\|\\ &\le -c_3\|x\|^2 + c_4\delta\|x\|\\ &= -(1-\theta)c_3\|x\|^2 - \theta c_3\|x\|^2 + c_4\delta\|x\|\\ &\le -(1-\theta)c_3\|x\|^2, \quad \forall\ \|x\| \ge \delta c_4/(\theta c_3) \stackrel{\text{def}}{=} \mu\end{aligned}$$

$$x(t_0) \in \Omega = \{V(x) \le c_1 r^2\}$$

$$\mu < r\sqrt{\frac{c_1}{c_2}} \ \Leftrightarrow\ \delta < \frac{c_3}{c_4}\sqrt{\frac{c_1}{c_2}}\theta r, \quad b = \mu\sqrt{\frac{c_2}{c_1}} \ \Leftrightarrow\ b = \frac{\delta c_4}{\theta c_3}\sqrt{\frac{c_2}{c_1}}$$

Example 4.10

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -4x_1 - 2x_2 + \beta x_2^3 + d(t)$$
$$\beta \geq 0, \qquad |d(t)| \leq \delta, \forall\ t \geq 0$$

$$V(x) = x^T P x = x^T \begin{bmatrix} \frac{3}{2} & \frac{1}{8} \\ \frac{1}{8} & \frac{5}{16} \end{bmatrix} x \quad \text{(Example 4.5)}$$

$$\begin{aligned} \dot{V}(t,x) &= -\|x\|^2 + 2\beta x_2^2 \left(\tfrac{1}{8}x_1 x_2 + \tfrac{5}{16}x_2^2\right) \\ &\quad + 2d(t)\left(\tfrac{1}{8}x_1 + \tfrac{5}{16}x_2\right) \\ &\leq -\|x\|^2 + \frac{\sqrt{29}}{8}\beta k_2^2 \|x\|^2 + \frac{\sqrt{29}\delta}{8}\|x\| \end{aligned}$$

$$k_2 = \max_{x^T P x \le c} |x_2| = 1.8194\sqrt{c}$$

Suppose $\beta \le 8(1-\zeta)/(\sqrt{29}k_2^2) \qquad (0 < \zeta < 1)$

$$\begin{aligned} \dot{V}(t,x) &\le -\zeta\|x\|^2 + \tfrac{\sqrt{29}\delta}{8}\|x\| \\ &\le -(1-\theta)\zeta\|x\|^2, \ \forall\ \|x\| \ge \tfrac{\sqrt{29}\delta}{8\zeta\theta} \stackrel{\text{def}}{=} \mu \end{aligned}$$

$$(0 < \theta < 1)$$

If $\mu^2\lambda_{\max}(P) < c$, then all solutions of the perturbed system, starting in Ω_c, are uniformly ultimately bounded by

$$b = \frac{\sqrt{29}\delta}{8\zeta\theta}\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}$$

Lemma 4.4

Suppose that $\forall\, x \in B_r$, $\forall\, t \geq 0$

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \quad \frac{\partial V}{\partial x} f(x) \leq -\alpha_3(\|x\|)$$

$$\left\| \frac{\partial V}{\partial x}(x) \right\| \leq k, \quad \|g(t,x)\| \leq \delta < \frac{\theta \alpha_3(\alpha_2^{-1}(\alpha_1(r)))}{k}$$

$\alpha_i \in \mathcal{K}$, $0 < \theta < 1$. Then, $\forall\, x(t_0) \in \{V(x) \leq \alpha_1(r)\}$

$$\|x(t)\| \leq \max\left\{\beta(\|x(t_0)\|, t - t_0),\ \rho(\delta)\right\}, \quad \forall\, t \geq t_0,\ \beta \in \mathcal{KL}$$

$$\rho(\delta) = \alpha_1^{-1}\left(\alpha_2\left(\alpha_3^{-1}\left(\frac{\delta k}{\theta}\right)\right)\right)$$

Proof: Apply Theorem 4.4

Compare

$$\text{Case 1:} \quad \delta < \frac{c_3}{c_4}\sqrt{\frac{c_1}{c_2}}\theta r$$

$$\text{Case 2:} \quad \delta < \frac{\theta\alpha_3(\alpha_2^{-1}(\alpha_1(r)))}{k}$$

Example 4.11 (Read)

$$\dot{x} = -\frac{x}{1+x^2} \quad \text{(Globally asymptotically stable)}$$

$$V(x) = x^4 \quad \Rightarrow \quad \frac{\partial V}{\partial x}\left[-\frac{x}{1+x^2}\right] = -\frac{4x^4}{1+x^2}$$

$$\alpha_1(|x|) = \alpha_2(|x|) = |x|^4; \qquad \alpha_3(|x|) = \frac{4|x|^4}{1+|x|^2}; \qquad k = 4r^3$$

$$\frac{\theta\alpha_3(\alpha_2^{-1}(\alpha_1(r)))}{k} = \frac{\theta\alpha_3(r)}{k} = \frac{r\theta}{1+r^2} < \tfrac{1}{2}$$

$$\dot{x} = -\frac{x}{1+x^2} + \delta, \quad \delta > \tfrac{1}{2} \quad \Rightarrow \quad \lim_{t\to\infty} x(t) = \infty$$

Input-to-State Stability (ISS)

Definition 4.4

The system $\dot{x} = f(x,u)$ is input-to-state stable if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for any initial state $x(t_0)$ and any bounded input $u(t)$

$$\|x(t)\| \le \max\left\{\beta(\|x(t_0)\|, t-t_0),\ \gamma\left(\sup_{t_0\le\tau\le t}\|u(\tau)\|\right)\right\}$$

for all $t \ge t_0$

ISS of $\dot{x} = f(x,u)$ implies

- BIBS stability
- $x(t)$ is ultimately bounded by $\gamma\left(\sup_{t_0\le\tau\le t}\|u(\tau)\|\right)$
- $\lim_{t\to\infty} u(t) = 0 \quad \Rightarrow \lim_{t\to\infty} x(t) = 0$
- The origin of $\dot{x} = f(x,0)$ is GAS

Theorem 4.6

Let $V(x)$ be a continuously differentiable function

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$$

$$\frac{\partial V}{\partial x} f(x, u) \leq -W_3(x), \ \ \forall \ \|x\| \geq \rho(\|u\|) > 0$$

$\forall \ x \in R^n, \ u \in R^m$, where $\alpha_1, \ \alpha_2 \in \mathcal{K}_\infty$, $\rho \in \mathcal{K}$, and $W_3(x)$ is a continuous positive definite function. Then, the system $\dot{x} = f(x, u)$ is ISS with $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$

Proof

Let $\mu = \rho(sup_{\tau \geq t_0} \|u(\tau)\|)$; then

$$\frac{\partial V}{\partial x} f(x, u) \leq -W_3(x), \ \ \forall \ \|x\| \geq \mu$$

Apply Theorem 4.4

$$\|x(t)\| \leq \max\left\{\beta(\|x(t_0)\|, t - t_0), \alpha_1^{-1}(\alpha_2(\mu))\right\}$$

$$\|x(t)\| \leq \max\left\{\beta(\|x(t_0)\|, t - t_0), \gamma\left(\sup_{\tau \geq t_0} \|u(\tau)\|\right)\right\}$$

Since $x(t)$ depends only on $u(\tau)$ for $t_0 \leq \tau \leq t$, the supremum on the right-hand side can be taken over $[t_0, t]$

Lemma 4.5

Suppose $f(x,u)$ is continuously differentiable and globally Lipschitz in (x,u). If $\dot{x} = f(x,0)$ has a globally exponentially stable equilibrium point at the origin, then the system $\dot{x} = f(x,u)$ is input-to-state stable

Proof: Apply (the converse Lyapunov) Theorem 3.8

Example 4.12

$$\dot{x} = -x^3 + u$$

The origin of $\dot{x} = -x^3$ is globally asymptotically stable

$$V = \tfrac{1}{2}x^2$$

$$\begin{aligned} \dot{V} &= -x^4 + xu \\ &= -(1-\theta)x^4 - \theta x^4 + xu \\ &\le -(1-\theta)x^4, \quad \forall\ |x| \ge \left(\tfrac{|u|}{\theta}\right)^{1/3} \end{aligned}$$

$$0 < \theta < 1$$

The system is ISS with $\gamma(r) = (r/\theta)^{1/3}$

Example 4.13

$$\dot{x} = -x - 2x^3 + (1 + x^2)u^2$$

The origin of $\dot{x} = -x - 2x^3$ is globally exponentially stable

$$V = \tfrac{1}{2}x^2$$

$$\begin{aligned} \dot{V} &= -x^2 - 2x^4 + x(1 + x^2)u^2 \\ &= -x^4 - x^2(1 + x^2) + x(1 + x^2)u^2 \\ &\leq -x^4, \quad \forall\ |x| \geq u^2 \end{aligned}$$

The system is ISS with $\gamma(r) = r^2$

Example 4.14

$$\dot{x}_1 = -x_1 + x_2, \qquad \dot{x}_2 = -x_1^3 - x_2 + u$$

Investigate GAS of $\qquad \dot{x}_1 = -x_1 + x_2, \qquad \dot{x}_2 = -x_1^3 - x_2$

$$V(x) = \tfrac{1}{4}x_1^4 + \tfrac{1}{2}x_2^2 \quad \Rightarrow \quad \dot{V} = -x_1^4 - x_2^2$$

Now $u \neq 0$

$$\dot{V} = -x_1^4 - x_2^2 + x_2 u \leq -x_1^4 - x_2^2 + |x_2|\;|u|$$

$$\dot{V} \leq -(1-\theta)[x_1^4 + x_2^2] - \theta x_1^4 - \theta x_2^2 + |x_2|\;|u|$$

$$(0 < \theta < 1)$$

$-\theta x_2^2 + |x_2|\ |u| \leq 0$ for $|x_2| \geq |u|/\theta$ and has a maximum value of $u^2/(4\theta)$ for $|x_2| < |u|/\theta$

$$x_1^2 \geq \frac{|u|}{2\theta} \ \text{ or } \ x_2^2 \geq \frac{u^2}{\theta^2} \quad \Rightarrow \quad -\theta x_1^4 - \theta x_2^2 + |x_2|\ |u| \leq 0$$

$$\|x\|^2 \geq \frac{|u|}{2\theta} + \frac{u^2}{\theta^2} \quad \Rightarrow \quad -\theta x_1^4 - \theta x_2^2 + |x_2|\ |u| \leq 0$$

$$\rho(r) = \sqrt{\frac{r}{2\theta} + \frac{r^2}{\theta^2}}$$

$$\dot{V} \leq -(1-\theta)[x_1^4 + x_2^2], \quad \forall\ \|x\| \geq \rho(|u|)$$

The system is ISS

Lemma 4.6

If the systems $\dot{\eta} = f_1(\eta, \xi)$ and $\dot{\xi} = f_2(\xi, u)$ are input-to-state stable, then the cascade connection

$$\dot{\eta} = f_1(\eta, \xi), \qquad \dot{\xi} = f_2(\xi, u)$$

is input-to-state stable. Consequently, If $\dot{\eta} = f_1(\eta, \xi)$ is input-to-state stable and the origin of $\dot{\xi} = f_2(\xi)$ is globally asymptotically stable, then the origin of the cascade connection

$$\dot{\eta} = f_1(\eta, \xi), \qquad \dot{\xi} = f_2(\xi)$$

is globally asymptotically stable

Example 4.15

$$\dot{x}_1 = -x_1 + x_2^2, \qquad \dot{x}_2 = -x_2 + u$$

The system $\dot{x}_1 = -x_1 + x_2^2$ is input-to-state stable, as seen from Theorem 4.6 with $V(x_1) = \frac{1}{2}x_1^2$

$$\dot{V} = -x_1^2 + x_1 x_2^2 \le -(1-\theta)x_1^2, \text{ for } |x_1| \ge x_2^2/\theta, \ 0 < \theta < 1$$

The linear system $\dot{x}_2 = -x_2 + u$ is input-to-state stable by Lemma 4.5

The cascade connection is input-to-state stable

Read

Definition 4.5

Let $\mathcal{X} \subset R^n$ and $\mathcal{U} \subset R^m$ be bounded sets containing their respective origins as interior points. The system $\dot{x} = f(x, u)$ is regionally input-to-state stable with respect to $\mathcal{X} \times \mathcal{U}$ if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for any initial state $x(t_0) \in \mathcal{X}$ and any input u with $u(t) \in \mathcal{U}$ for all $t \geq t_0$, the solution $x(t)$ belongs to $\mathcal{X}$ for all $t \geq t_0$ and satisfies

$$\|x(t)\| \leq \max\left\{\beta(\|x(t_0)\|, t - t_0),\ \gamma\left(\sup_{t_0 \leq \tau \leq t} \|u(\tau)\|\right)\right\}$$

The system $\dot{x} = f(x, u)$ is locally input-to-state stable if it is regionally input-to-state stable with respect to some neighborhood of the origin $(x = 0, u = 0)$

Theorem 4.7

Suppose $f(x, u)$ is locally Lipschitz in (x, u) for all $x \in B_r$ and $u \in B_\lambda$. Let $V(x)$ be a continuously differentiable function that satisfies

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$$

$$\frac{\partial V}{\partial x} f(x, u) \leq -W_3(x), \quad \forall \; \|x\| \geq \rho(\|u\|) > 0$$

for all $x \in B_r$ and $u \in B_\lambda$, where α_1, α_2, $\rho \in \mathcal{K}$ and $W_3(x)$ is a continuous positive definite function. Suppose $\alpha_1(r) > \alpha_2(\rho(\lambda))$ and let $\Omega = \{V(x) \leq \alpha_1(r)\}$. Then, the system $\dot{x} = f(x, u)$ is regionally input-to-state stable with respect to $\Omega \times B_\lambda$ and $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$

Local input-to-state stability of $\dot{x} = f(x, u)$ is equivalent to asymptotic stability of the origin of $\dot{x} = f(x, 0)$

Lemma 4.7

Suppose $f(x, u)$ is locally Lipschitz in (x, u) in some neighborhood of $(x = 0, u = 0)$. Then, the system $\dot{x} = f(x, u)$ is locally input-to-state stable if and only if the unforced system $\dot{x} = f(x, 0)$ has an asymptotically stable equilibrium point at the origin

The proof uses (converse Lyapunov) Theorem 3.9