Nonlinear Control

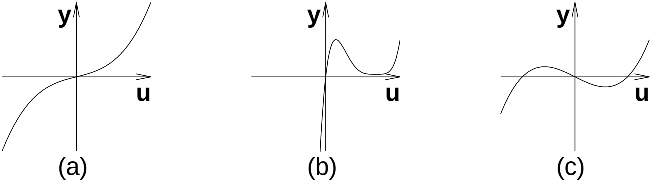
Lecture # 6

Passivity

and

Input-Output Stability

Passivity: Memoryless Functions



Passive   Passive   Not passive

$$y = h(t, u), \qquad h \in [0, \infty]$$

Vector case: $y = h(t, u), \qquad h^T = \left[\ h_1, \ \ h_2, \ \ \cdots, \ \ h_p\ \right]$

power inflow $= \Sigma_{i=1}^{p} u_i y_i = \ u^T y$

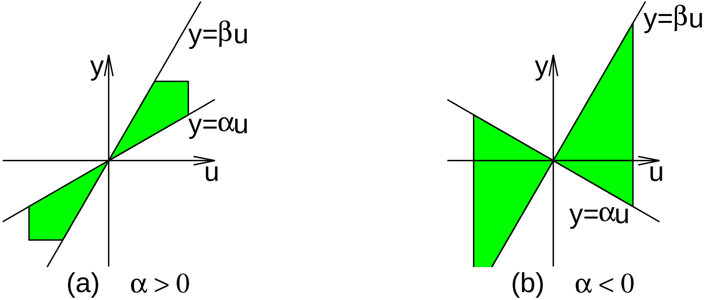
Definition 5.1

$y = h(t, u)$ is

- passive if $u^T y \geq 0$
- lossless if $u^T y = 0$
- input strictly passive if $u^T y \geq u^T \varphi(u)$ for some function φ where $u^T \varphi(u) > 0$, $\forall\ u \neq 0$
- output strictly passive if $u^T y \geq y^T \rho(y)$ for some function ρ where $y^T \rho(y) > 0$, $\forall\ y \neq 0$

Sector Nonlinearity: h belongs to the sector $[\alpha, \beta]$ ($h \in [\alpha, \beta]$) if

$$\alpha u^2 \leq u h(t, u) \leq \beta u^2$$



Also, $h \in (\alpha, \beta]$, $h \in [\alpha, \beta)$, $h \in (\alpha, \beta)$

$$\alpha u^2 \leq uh(t,u) \leq \beta u^2 \Leftrightarrow [h(t,u) - \alpha u][h(t,u) - \beta u] \leq 0$$

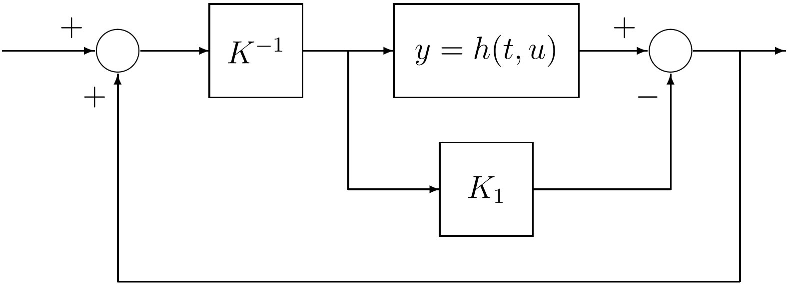
Definition 5.2

A memoryless function $h(t,u)$ is said to belong to the sector

- $[0, \infty]$ if $u^T h(t,u) \geq 0$
- $[K_1, \infty]$ if $u^T [h(t,u) - K_1 u] \geq 0$
- $[0, K_2]$ with $K_2 = K_2^T > 0$ if $h^T(t,u)[h(t,u) - K_2 u] \leq 0$
- $[K_1, K_2]$ with $K = K_2 - K_1 = K^T > 0$ if

$$[h(t,u) - K_1 u]^T [h(t,u) - K_2 u] \leq 0$$

A function in the sector $[K_1, K_2]$ can be transformed into a function in the sector $[0, \infty]$ by input feedforward followed by output feedback



$$[K_1, K_2] \xrightarrow{\text{Feedforward}} [0, K] \xrightarrow{K^{-1}} [0, I] \xrightarrow{\text{Feedback}} [0, \infty]$$

Passivity: State Models

Definition 5.3

The system

$$\dot{x} = f(x,u), \qquad y = h(x,u)$$

is passive if there is a continuously differentiable positive semidefinite function $V(x)$ (the storage function) such that

$$u^T y \geq \dot{V} = \frac{\partial V}{\partial x} f(x,u), \ \ \forall \ (x,u)$$

Moreover, it is

- lossless if $u^T y = \dot{V}$
- input strictly passive if $u^T y \geq \dot{V} + u^T \varphi(u)$ for some function φ such that $u^T \varphi(u) > 0, \ \forall \ u \neq 0$
- output strictly passive if $u^T y \geq \dot{V} + y^T \rho(y)$ for some function ρ such that $y^T \rho(y) > 0, \ \forall \ y \neq 0$
- strictly passive if $u^T y \geq \dot{V} + \psi(x)$ for some positive definite function ψ

Example 5.2

$$\dot{x} = u, \qquad y = x$$

$$V(x) = \tfrac{1}{2}x^2 \quad \Rightarrow \quad uy = \dot{V} \quad \Rightarrow \quad \text{Lossless}$$

$$\dot{x} = u, \qquad y = x + h(u), \qquad h \in [0, \infty]$$

$$V(x) = \tfrac{1}{2}x^2 \quad \Rightarrow \quad uy = \dot{V} + uh(u) \quad \Rightarrow \quad \text{Passive}$$

$$h \in (0, \infty] \quad \Rightarrow \quad uh(u) > 0 \;\forall\; u \neq 0$$

$$\Rightarrow \quad \text{Input strictly passive}$$

$$\dot{x} = -h(x) + u, \qquad y = x, \qquad h \in [0, \infty]$$

$$V(x) = \tfrac{1}{2}x^2 \quad \Rightarrow \quad uy = \dot{V} + yh(y) \quad \Rightarrow \quad \text{Passive}$$

$$h \in (0, \infty] \quad \Rightarrow \quad \text{Output strictly passive}$$

Example 5.3

$$\dot{x} = u, \qquad y = h(x), \qquad h \in [0, \infty]$$

$$V(x) = \int_0^x h(\sigma)\ d\sigma \quad \Rightarrow \quad \dot{V} = h(x)\dot{x} = yu \quad \Rightarrow \quad \text{Lossless}$$

$$a\dot{x} = -x + u, \qquad y = h(x), \qquad h \in [0, \infty]$$

$$V(x) = a\int_0^x h(\sigma)\ d\sigma \quad \Rightarrow \quad \dot{V} = h(x)(-x + u) = yu - xh(x)$$

$$yu = \dot{V} + xh(x) \quad \Rightarrow \quad \text{Passive}$$

$$h \in (0, \infty] \quad \Rightarrow \quad \text{Strictly passive}$$

Example 5.4

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -h(x_1) - ax_2 + u, \qquad y = bx_2 + u$$

$$h \in [\alpha_1, \infty], \quad a > 0, \quad b > 0, \quad \alpha_1 > 0$$

$$\begin{aligned} V(x) &= \alpha \int_0^{x_1} h(\sigma)\ d\sigma + \tfrac{1}{2}\alpha x^T P x \\ &= \alpha \int_0^{x_1} h(\sigma)\ d\sigma + \tfrac{1}{2}\alpha(p_{11}x_1^2 + 2p_{12}x_1x_2 + p_{22}x_2^2) \end{aligned}$$

$$\alpha > 0, \quad p_{11} > 0, \quad p_{11}p_{22} - p_{12}^2 > 0$$

$$
\begin{aligned}
uy - \dot{V} &= u(bx_2 + u) - \alpha[h(x_1) + p_{11}x_1 + p_{12}x_2]x_2 \\
&\quad - \alpha(p_{12}x_1 + p_{22}x_2)[-h(x_1) - ax_2 + u]
\end{aligned}
$$

Take $p_{22} = 1$, $p_{11} = ap_{12}$, and $\alpha = b$ to cancel the cross product terms

$$uy - \dot{V} \geq bp_{12}\left(\alpha_1 - \tfrac{1}{4}bp_{12}\right)x_1^2 + b(a - p_{12})x_2^2$$

$$p_{12} = ak, \quad 0 < k < \min\{1, 4\alpha_1/(ab)\}$$

$$\Rightarrow p_{11} > 0, \; p_{11}p_{22} - p_{12}^2 > 0$$

$$\Rightarrow \; bp_{12}\left(\alpha_1 - \tfrac{1}{4}bp_{12}\right) > 0, \; b(a - p_{12}) > 0$$

$\Rightarrow$ Strictly passive

Positive Real Transfer Functions

Definition 5.4

An $m \times m$ proper rational transfer function matrix $G(s)$ is positive real if

- poles of all elements of $G(s)$ are in $Re[s] \leq 0$
- for all real ω for which $j\omega$ is not a pole of any element of $G(s)$, the matrix $G(j\omega) + G^T(-j\omega)$ is positive semidefinite
- any pure imaginary pole $j\omega$ of any element of $G(s)$ is a simple pole and the residue matrix $\lim_{s \to j\omega}(s - j\omega)G(s)$ is positive semidefinite Hermitian

$G(s)$ is strictly positive real if $G(s - \varepsilon)$ is positive real for some $\varepsilon > 0$

Scalar Case ($m = 1$):

$$G(j\omega) + G^T(-j\omega) = 2Re[G(j\omega)]$$

$Re[G(j\omega)]$ is an even function of ω. The second condition of the definition reduces to

$$Re[G(j\omega)] \geq 0, \ \forall \ \omega \in [0, \infty)$$

which holds when the Nyquist plot of of $G(j\omega)$ lies in the closed right-half complex plane

This is true only if the relative degree of the transfer function is zero or one

Lemma 5.1

An $m \times m$ proper rational transfer function matrix $G(s)$ is strictly positive real if and only if

- $G(s)$ is Hurwitz
- $G(j\omega) + G^T(-j\omega) > 0, \ \forall \ \omega \in R$
- $G(\infty) + G^T(\infty) > 0$ or

$$\lim_{\omega \to \infty} \omega^{2(m-q)} \det[G(j\omega) + G^T(-j\omega)] > 0$$

where $q = \text{rank}[G(\infty) + G^T(\infty)]$

Scalar Case ($m = 1$): $G(s)$ is strictly positive real if and only if

- $G(s)$ is Hurwitz
- $Re[G(j\omega)] > 0, \ \forall \ \omega \in [0, \infty)$
- $G(\infty) > 0$ or

$$\lim_{\omega \to \infty} \omega^2 Re[G(j\omega)] > 0$$

Positive Real Lemma (5.2)

Let

$$G(s) = C(sI - A)^{-1}B + D$$

where (A, B) is controllable and (A, C) is observable. $G(s)$ is positive real if and only if there exist matrices $P = P^T > 0$, L, and W such that

$$\begin{aligned} PA + A^TP &= -L^TL \\ PB &= C^T - L^TW \\ W^TW &= D + D^T \end{aligned}$$

Kalman–Yakubovich–Popov Lemma (5.3)

Let

$$G(s) = C(sI - A)^{-1}B + D$$

where (A, B) is controllable and (A, C) is observable. $G(s)$ is strictly positive real if and only if there exist matrices $P = P^T > 0$, L, and W, and a positive constant ε such that

$$\begin{aligned} PA + A^T P &= -L^T L - \varepsilon P \\ PB &= C^T - L^T W \\ W^T W &= D + D^T \end{aligned}$$

Lemma 5.4

The linear time-invariant minimal realization

$$\dot{x} = Ax + Bu, \qquad y = Cx + Du$$

with

$$G(s) = C(sI - A)^{-1}B + D$$

is

- passive if $G(s)$ is positive real
- strictly passive if $G(s)$ is strictly positive real

Proof

Apply the PR and KYP Lemmas, respectively, and use $V(x) = \frac{1}{2}x^T Px$ as the storage function

Connection with Stability

Lemma 5.5

If the system

$$\dot{x} = f(x,u), \qquad y = h(x,u)$$

is passive with a positive definite storage function $V(x)$, then the origin of $\dot{x} = f(x,0)$ is stable

Proof

$$u^T y \geq \frac{\partial V}{\partial x} f(x,u) \;\; \Rightarrow \;\; \frac{\partial V}{\partial x} f(x,0) \leq 0$$

Lemma 5.6

If the system

$$\dot{x} = f(x, u), \qquad y = h(x, u)$$

is strictly passive, then the origin of $\dot{x} = f(x, 0)$ is asymptotically stable. Furthermore, if the storage function is radially unbounded, the origin will be globally asymptotically stable

Proof

The storage function $V(x)$ is positive definite

$$u^T y \geq \frac{\partial V}{\partial x} f(x, u) + \psi(x) \quad \Rightarrow \quad \frac{\partial V}{\partial x} f(x, 0) \leq -\psi(x)$$

Definition 5.5

The system

$$\dot{x} = f(x, u), \qquad y = h(x, u)$$

is zero-state observable if no solution of $\dot{x} = f(x, 0)$ can stay identically in $S = \{h(x, 0) = 0\}$, other than the zero solution $x(t) \equiv 0$

Linear Systems

$$\dot{x} = Ax, \quad y = Cx$$

Observability of (A, C) is equivalent to

$$y(t) = Ce^{At}x(0) \equiv 0 \;\Leftrightarrow\; x(0) = 0 \;\Leftrightarrow\; x(t) \equiv 0$$

Lemma 5.6

If the system

$$\dot{x} = f(x, u), \qquad y = h(x, u)$$

is output strictly passive and zero-state observable, then the origin of $\dot{x} = f(x, 0)$ is asymptotically stable. Furthermore, if the storage function is radially unbounded, the origin will be globally asymptotically stable

Proof

The storage function $V(x)$ is positive definite

$$u^T y \geq \frac{\partial V}{\partial x} f(x, u) + y^T \rho(y) \;\; \Rightarrow \;\; \frac{\partial V}{\partial x} f(x, 0) \leq -y^T \rho(y)$$

$$\dot{V}(x(t)) \equiv 0 \; \Rightarrow \; y(t) \equiv 0 \; \Rightarrow x(t) \equiv 0$$

Apply the invariance principle

Example 5.8

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -ax_1^3 - kx_2 + u, \quad y = x_2, \quad a, k > 0$$

$$V(x) = \tfrac{1}{4}ax_1^4 + \tfrac{1}{2}x_2^2$$

$$\dot{V} = ax_1^3 x_2 + x_2(-ax_1^3 - kx_2 + u) = -ky^2 + yu$$

The system is output strictly passive

$$y(t) \equiv 0 \;\Leftrightarrow\; x_2(t) \equiv 0 \;\Rightarrow\; ax_1^3(t) \equiv 0 \;\Rightarrow\; x_1(t) \equiv 0$$

The system is zero-state observable. V is radially unbounded. Hence, the origin of the unforced system is globally asymptotically stable

$\mathcal{L}$ Stability

Input-Output Models: $\quad y = Hu$

$u(t)$ is a piecewise continuous function of t and belongs to a linear space of signals

- The space of bounded functions: $\sup_{t\geq 0} \|u(t)\| < \infty$
- The space of square-integrable functions: $\int_0^\infty u^T(t)u(t)\ dt < \infty$

Norm of a signal $\|u\|$:

- $\|u\| \geq 0$ and $\|u\| = 0 \quad \Leftrightarrow \quad u = 0$
- $\|au\| = a\|u\|$ for any $a > 0$
- Triangle Inequality: $\|u_1 + u_2\| \leq \|u_1\| + \|u_2\|$

$\mathcal{L}_p$ spaces:

$$\mathcal{L}_\infty : \quad \|u\|_{\mathcal{L}_\infty} = \sup_{t \geq 0} \|u(t)\| < \infty$$

$$\mathcal{L}_2 : \quad \|u\|_{\mathcal{L}_2} = \sqrt{\int_0^\infty u^T(t) u(t) \; dt} < \infty$$

$$\mathcal{L}_p : \quad \|u\|_{\mathcal{L}_p} = \left(\int_0^\infty \|u(t)\|^p \; dt \right)^{1/p} < \infty, \quad 1 \leq p < \infty$$

Notation $\mathcal{L}_p^m$: p is the type of p-norm used to define the space and m is the dimension of u

Extended Space: $\mathcal{L}_e = \{u \mid u_\tau \in \mathcal{L}, \forall\ \tau \in [0,\infty)\}$

u_τ is a truncation of u: $u_\tau(t) = \begin{cases} u(t), & 0 \le t \le \tau \\ 0, & t > \tau \end{cases}$

$\mathcal{L}_e$ is a linear space and $\mathcal{L} \subset \mathcal{L}_e$

Example

$$u(t) = t, \qquad u_\tau(t) = \begin{cases} t, & 0 \le t \le \tau \\ 0, & t > \tau \end{cases}$$

$u \notin \mathcal{L}_\infty$ but $u_\tau \in \mathcal{L}_{\infty e}$

Causality: A mapping $H : \mathcal{L}_e^m \to \mathcal{L}_e^q$ is causal if the value of the output $(Hu)(t)$ at any time t depends only on the values of the input up to time t

$$(Hu)_\tau = (Hu_\tau)_\tau$$

Definition 6.1

A scalar continuous function $g(r)$, defined for $r \in [0, a)$, is a gain function if it is nondecreasing and $g(0) = 0$

A class $\mathcal{K}$ function is a gain function but not the other way around. By not requiring the gain function to be strictly increasing we can have $g = 0$ or $g(r) = \mathrm{sat}(r)$

Definition 6.2

A mapping $H : \mathcal{L}_e^m \to \mathcal{L}_e^q$ is $\mathcal{L}$ stable if there exist a gain function g, defined on $[0, \infty)$, and a nonnegative constant β such that

$$\|(Hu)_\tau\|_{\mathcal{L}} \le g\left(\|u_\tau\|_{\mathcal{L}}\right) + \beta, \quad \forall\ u \in \mathcal{L}_e^m \text{ and } \tau \in [0, \infty)$$

It is finite-gain $\mathcal{L}$ stable if there exist nonnegative constants γ and β such that

$$\|(Hu)_\tau\|_{\mathcal{L}} \le \gamma\|u_\tau\|_{\mathcal{L}} + \beta, \quad \forall\ u \in \mathcal{L}_e^m \text{ and } \tau \in [0, \infty)$$

In this case, we say that the system has $\mathcal{L}$ gain $\le \gamma$. The bias term β is included in the definition to allow for systems where Hu does not vanish at $u = 0$.

$\mathcal{L}$ Stability of State Models

$$\dot{x} = f(x,u), \quad y = h(x,u), \qquad 0 = f(0,0), \ 0 = h(0,0)$$

Case 1: The origin of $\dot{x} = f(x,0)$ is exponentially stable

Theorem 6.1

Suppose, $\forall \ \|x\| \leq r$, $\forall \ \|u\| \leq r_u$,

$$c_1\|x\|^2 \leq V(x) \leq c_2\|x\|^2$$

$$\frac{\partial V}{\partial x} f(x,0) \leq -c_3\|x\|^2, \quad \left\|\frac{\partial V}{\partial x}\right\| \leq c_4\|x\|$$

$$\|f(x,u) - f(x,0)\| \leq L\|u\|, \quad \|h(x,u)\| \leq \eta_1\|x\| + \eta_2\|u\|$$

Then, for each x_0 with $\|x_0\| \leq r\sqrt{c_1/c_2}$, the system is small-signal finite-gain $\mathcal{L}_p$ stable for each $p \in [1,\infty]$. It is finite-gain $\mathcal{L}_p$ stable $\forall \ x_0 \in R^n$ if the assumptions hold globally [see the textbook for β and γ]

Example 6.4

$$\dot{x} = -x - x^3 + u, \qquad y = \tanh x + u$$

$$V = \tfrac{1}{2}x^2 \quad \Rightarrow \quad \dot{V} = x(-x - x^3) \le -x^2$$

$$c_1 = c_2 = \tfrac{1}{2}, \; c_3 = c_4 = 1, \quad L = \eta_1 = \eta_2 = 1$$

Finite-gain $\mathcal{L}_p$ stable for each $x(0) \in R$ and each $p \in [1, \infty]$

Example 6.5

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 - x_2 - a \tanh x_1 + u, \quad y = x_1, \quad a \ge 0$$

$$V(x) = x^T P x = p_{11}x_1^2 + 2p_{12}x_1x_2 + p_{22}x_2^2$$

$$\dot{V} = -2p_{12}(x_1^2 + ax_1 \tanh x_1) + 2(p_{11} - p_{12} - p_{22})x_1x_2 - 2ap_{22}x_2 \tanh x_1 - 2(p_{22} - p_{12})x_2^2$$

$p_{11} = p_{12} + p_{22} \quad \Rightarrow \quad$ the term x_1x_2 is canceled

$p_{22} = 2p_{12} = 1 \quad \Rightarrow \quad$ P is positive definite

$$\dot{V} = -x_1^2 - x_2^2 - ax_1 \tanh x_1 - 2ax_2 \tanh x_1$$

$$\dot{V} \le -\|x\|^2 + 2a|x_1|\ |x_2| \le -(1-a)\|x\|^2$$

$$a < 1 \Rightarrow c_1 = \lambda_{\min}(P), c_2 = \lambda_{\max}(P), c_3 = 1 - a, c_4 = 2c_2$$

$$L = \eta_1 = 1, \quad \eta_2 = 0$$

For each $x(0) \in R^2,\ p \in [1, \infty]$, the system is finite-gain $\mathcal{L}_p$ stable

$$\gamma = 2[\lambda_{\max}(P)]^2/[(1-a)\lambda_{\min}(P)]$$

Case 2: The origin of $\dot{x} = f(x, 0)$ is asymptotically stable

Theorem 6.2

Suppose that, for all (x, u), f is locally Lipschitz and h is continuous and satisfies

$$\|h(x, u)\| \leq g_1(\|x\|) + g_2(\|u\|) + \eta, \quad \eta \geq 0$$

for some gain functions g_1, g_2. If $\dot{x} = f(x, u)$ is ISS, then, for each $x(0) \in R^n$, the system

$$\dot{x} = f(x, u), \qquad y = h(x, u)$$

is $\mathcal{L}_\infty$ stable

Example 6.6

$$\dot{x} = -x - 2x^3 + (1 + x^2)u^2, \qquad y = x^2 + u$$

ISS from Example 4.13

$$g_1(r) = r^2, \quad g_2(r) = r, \quad \eta = 0$$

$\mathcal{L}_\infty$ stable

Example 6.7

$$\dot{x}_1 = -x_1^3 + x_2, \quad \dot{x}_2 = -x_1 - x_2^3 + u, \qquad y = x_1 + x_2$$

$$V = (x_1^2 + x_2^2) \quad \Rightarrow \quad \dot{V} = -2x_1^4 - 2x_2^4 + 2x_2 u$$

$$x_1^4 + x_2^4 \geq \tfrac{1}{2}\|x\|^4$$

$$\begin{aligned} \dot{V} &\leq -\|x\|^4 + 2\|x\||u| \\ &= -(1-\theta)\|x\|^4 - \theta\|x\|^4 + 2\|x\||u|, \quad 0 < \theta < 1 \\ &\leq -(1-\theta)\|x\|^4, \quad \forall\ \|x\| \geq \left(\frac{2|u|}{\theta}\right)^{1/3} \ \Rightarrow \ \text{ISS} \end{aligned}$$

$$g_1(r) = \sqrt{2}r, \quad g_2 = 0, \quad \eta = 0$$

$\mathcal{L}_\infty$ stable