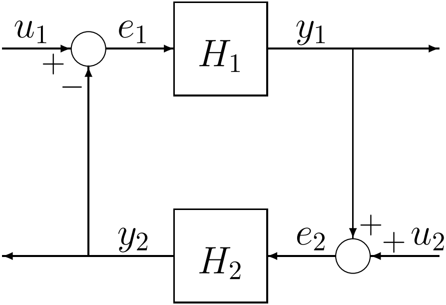
Nonlinear Control

Lecture # 7

Stability of Feedback Systems



$$\dot{x}_i = f_i(x_i, e_i), \qquad y_i = h_i(x_i, e_i)$$

$$y_i = h_i(t, e_i)$$

Passivity Theorems

Theorem 7.1

The feedback connection of two passive systems is passive

Proof

Let $V_1(x_1)$ and $V_2(x_2)$ be the storage functions for H_1 and H_2 ($V_i = 0$ if H_i is memoryless)

$$e_i^T y_i \geq \dot{V}_i, \qquad V(x) = V_1(x_1) + V_2(x_2)$$

$$e_1^T y_1 + e_2^T y_2 = (u_1 - y_2)^T y_1 + (u_2 + y_1)^T y_2 = u_1^T y_1 + u_2^T y_2$$

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \qquad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$u^T y = u_1^T y_1 + u_2^T y_2 \geq \dot{V}_1 + \dot{V}_2 = \dot{V}$$

Asymptotic Stability

Theorem 7.2

Consider the feedback connection of two dynamical systems. When $u = 0$, the origin of the closed-loop system is asymptotically stable if one of the following conditions is satisfied:

- both feedback components are strictly passive;
- both feedback components are output strictly passive and zero-state observable;
- one component is strictly passive and the other one is output strictly passive and zero-state observable.

If the storage function for each component is radially unbounded, the origin is globally asymptotically stable

Proof

H_1 is SP; H_2 is OSP & ZSO

$$e_1^T y_1 \geq \dot{V}_1 + \psi_1(x_1), \qquad \psi_1(x_1) > 0, \ \forall \ x_1 \neq 0$$

$$e_2^T y_2 \geq \dot{V}_2 + y_2^T \rho_2(y_2), \qquad y_2^T \rho(y_2) > 0, \ \forall y_2 \neq 0$$

$$e_1^T y_1 + e_2^T y_2 = (u_1 - y_2)^T y_1 + (u_2 + y_1)^T y_2 = u_1^T y_1 + u_2^T y_2$$

$$V(x) = V_1(x_1) + V_2(x_2)$$

$$\dot{V} \leq u^T y - \psi_1(x_1) - y_2^T \rho_2(y_2)$$

$$u = 0 \quad \Rightarrow \quad \dot{V} \leq -\psi_1(x_1) - y_2^T \rho_2(y_2)$$

$$\dot{V} \leq -\psi_1(x_1) - y_2^T \rho_2(y_2)$$

$$\dot{V} = 0 \;\;\Rightarrow\;\; x_1 = 0 \text{ and } y_2 = 0$$

$$y_2(t) \equiv 0 \;\Rightarrow\; e_1(t) \equiv 0 \;\; (\;\&\; x_1(t) \equiv 0) \;\Rightarrow\; y_1(t) \equiv 0$$

$$y_1(t) \equiv 0 \;\Rightarrow\; e_2(t) \equiv 0$$

By zero-state observability of H_2: $y_2(t) \equiv 0 \;\Rightarrow\; x_2(t) \equiv 0$

Apply the invariance principle

Example 7.1

$$\underbrace{\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -ax_1^3 - kx_2 + e_1 \\ y_1 &= x_2 \end{aligned}}_{H_1} \qquad \Bigg| \qquad \underbrace{\begin{aligned} \dot{x}_3 &= x_4 \\ \dot{x}_4 &= -bx_3 - x_4^3 + e_2 \\ y_2 &= x_4 \end{aligned}}_{H_2}$$

$$a, \; b, \; k > 0$$

$$V_1 = \tfrac{1}{4}ax_1^4 + \tfrac{1}{2}x_2^2$$

$$\dot{V}_1 = ax_1^3x_2 - ax_1^3x_2 - kx_2^2 + x_2e_1 = -ky_1^2 + y_1e_1$$

With $e_1 = 0, \quad y_1(t) \equiv 0 \;\Leftrightarrow\; x_2(t) \equiv 0 \;\Rightarrow\; x_1(t) \equiv 0$

H_1 is output strictly passive and zero-state observable

$$V_2 = \frac{1}{2}bx_3^2 + \frac{1}{2}x_4^2$$

$$\dot{V}_2 = bx_3x_4 - bx_3x_4 - x_4^4 + x_4e_2 = -y_2^4 + y_2e_2$$

With $e_2 = 0, \quad y_2(t) \equiv 0 \;\Leftrightarrow\; x_4(t) \equiv 0 \;\Rightarrow\; x_3(t) \equiv 0$

H_2 is output strictly passive and zero-state observable

V_1 and V_2 are radially unbounded

The origin is globally asymptotically stable

Theorem 7.3

Consider the feedback connection of a strictly passive dynamical system with a passive time-varying memoryless function. When $u = 0$, the origin of the closed-loop system is uniformly asymptotically stable. if the storage function for the dynamical system is radially unbounded, the origin will be globally uniformly asymptotically stable

Proof

Let $V_1(x_1)$ be (positive definite) storage function of H_1.

$$\dot{V}_1 = \frac{\partial V_1}{\partial x_1} f_1(x_1, e_1) \leq e_1^T y_1 - \psi_1(x_1) = -e_2^T y_2 - \psi_1(x_1)$$

$$e_2^T y_2 \geq 0 \quad \Rightarrow \quad \dot{V}_1 \leq -\psi_1(x_1)$$

Example 7.4

Consider the feedback connection of a strictly positive real transfer function and a passive time-varying memoryless function

From Lemma 5.4, we know that the dynamical system is strictly passive with a positive definite storage function $V(x) = \frac{1}{2}x^T Px$

From Theorem 7.3, the origin of the closed-loop system is globally uniformly asymptotically stable

Theorem 7.4

Consider the feedback connection of a time-invariant dynamical system H_1 with a time-invariant memoryless function H_2. Suppose H_1 is zero-state observable, $V_1(x_1)$ is positive definite

$$e_1^T y_1 \geq \dot{V}_1 + y_1^T \rho_1(y_1), \qquad e_2^T y_2 \geq e_2^T \varphi_2(e_2)$$

Then, the origin of the closed-loop system (when $u = 0$) is asymptotically stable if

$$v^T[\rho_1(v) + \varphi_2(v)] > 0, \quad \forall\ v \neq 0$$

Furthermore, if V_1 is radially unbounded, the origin will be globally asymptotically stable

Example 7.5

$$\underbrace{\begin{array}{rcl} \dot{x} & = & f(x) + G(x)e_1 \\ y_1 & = & h(x) \end{array}}_{H_1} \quad \Bigg| \quad \underbrace{y_2 = \sigma(e_2)}_{H_2}$$

$$\sigma(0) = 0, \quad e_2^T \sigma(e_2) > 0, \ \forall \ e_2 \neq 0$$

Suppose H_1 is zero-state observable and there is a radially unbounded positive definite function $V_1(x)$ such that

$$\frac{\partial V_1}{\partial x} f(x) \leq 0, \qquad \frac{\partial V_1}{\partial x} G(x) = h^T(x), \quad \forall \ x \in R^n$$

$$\dot{V}_1 = \frac{\partial V_1}{\partial x} f(x) + \frac{\partial V_1}{\partial x} G(x) e_1 \leq y_1^T e_1$$

Apply Theorem 7.4:

$$\dot{V}_1 \le e_1^T y_1$$

$$e_1^T y_1 \ge \dot{V}_1 + y_1^T \rho_1(y_1) \quad \text{is satisfied with} \quad \rho_1 = 0$$

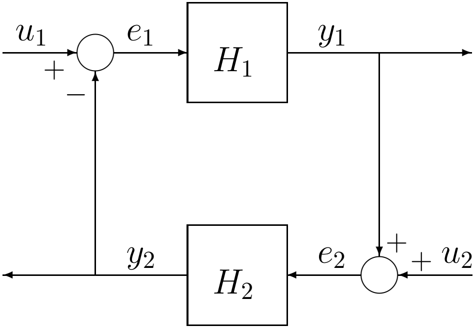
$$e_2^T y_2 = e_2^T \sigma(e_2)$$

$$e_2^T y_2 \ge e_2^T \varphi_2(e_2) \quad \text{is satisfied with} \quad \varphi_2 = \sigma$$

$$v^T[\rho_1(v) + \varphi_2(v)] = v^T \sigma(v) > 0, \quad \forall\ v \ne 0$$

The origin is globally asymptotically stable

The Small-Gain Theorem



$$\|y_{1\tau}\|_{\mathcal{L}} \leq \gamma_1 \|e_{1\tau}\|_{\mathcal{L}} + \beta_1, \quad \forall\ e_1 \in \mathcal{L}_e^m, \ \forall\ \tau \in [0, \infty)$$

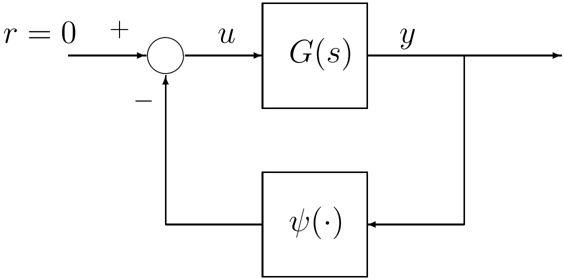
$$\|y_{2\tau}\|_{\mathcal{L}} \leq \gamma_2 \|e_{2\tau}\|_{\mathcal{L}} + \beta_2, \quad \forall\ e_2 \in \mathcal{L}_e^q, \ \forall\ \tau \in [0, \infty)$$

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

Theorem 7.7

The feedback connection is finite-gain $\mathcal{L}$ stable if $\gamma_1\gamma_2 < 1$

Absolute Stability



Definition 7.1

The system is absolutely stable if the origin is globally uniformly asymptotically stable for any nonlinearity in a given sector. It is absolutely stable with finite domain if the origin is uniformly asymptotically stable

Circle Criterion

Suppose $G(s) = C(sI - A)^{-1}B + D$ is SPR, $\psi \in [0, \infty]$

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \\ u &= -\psi(t, y) \end{aligned}$$

By the KYP Lemma, $\exists\ P = P^T > 0,\ L, W, \varepsilon > 0$

$$\begin{aligned} PA + A^TP &= -L^TL - \varepsilon P \\ PB &= C^T - L^TW \\ W^TW &= D + D^T \end{aligned}$$

$$V(x) = \tfrac{1}{2}x^TPx$$

$$
\begin{aligned}
\dot{V} &= \tfrac{1}{2}x^T P\dot{x} + \tfrac{1}{2}\dot{x}^T P x \\
&= \tfrac{1}{2}x^T(PA + A^T P)x + x^T PBu \\
&= -\tfrac{1}{2}x^T L^T Lx - \tfrac{1}{2}\varepsilon x^T Px + x^T(C^T - L^T W)u \\
&= -\tfrac{1}{2}x^T L^T Lx - \tfrac{1}{2}\varepsilon x^T Px + (Cx + Du)^T u \\
&\quad - u^T Du - x^T L^T Wu
\end{aligned}
$$

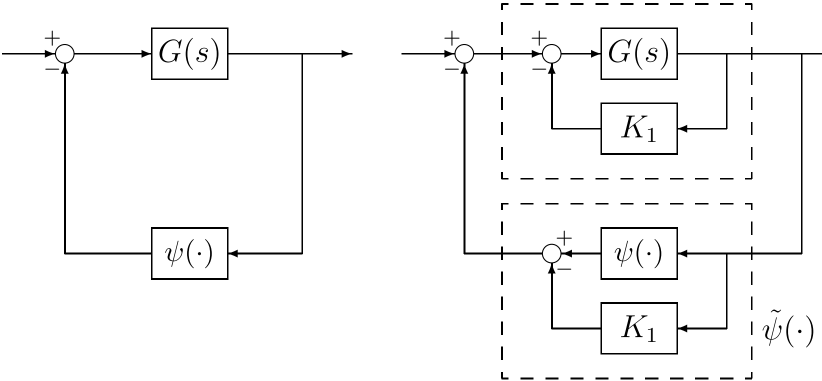
$$u^T Du = \tfrac{1}{2}u^T(D + D^T)u = \tfrac{1}{2}u^T W^T Wu$$

$$\dot{V} = -\tfrac{1}{2}\varepsilon x^T Px - \tfrac{1}{2}(Lx + Wu)^T(Lx + Wu) - y^T\psi(t, y)$$

$$y^T\psi(t, y) \geq 0 \quad \Rightarrow \quad \dot{V} \leq -\tfrac{1}{2}\varepsilon x^T Px$$

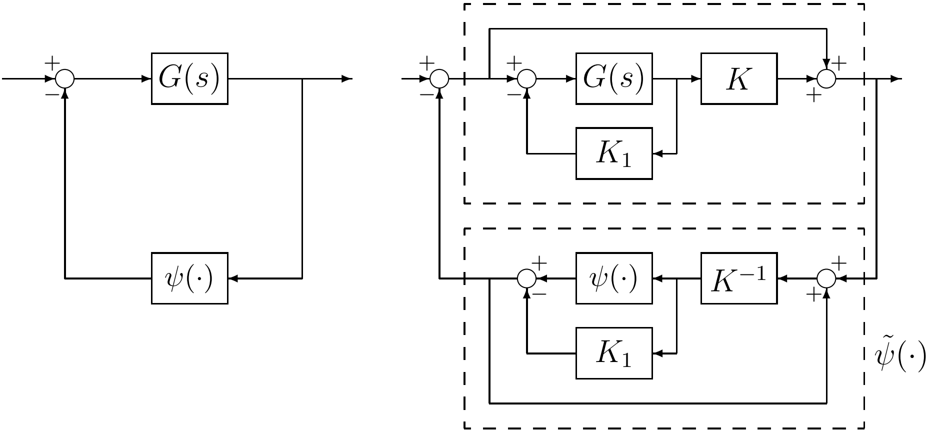
The origin is globally exponentially stable

What if $\psi \in [K_1, \infty]$?



$\tilde{\psi} \in [0, \infty]$; hence the origin is globally exponentially stable if $G(s)[I + K_1 G(s)]^{-1}$ is SPR

What if $\psi \in [K_1, K_2]$?



$\tilde{\psi} \in [0, \infty]$; hence the origin is globally exponentially stable if $I + KG(s)[I + K_1 G(s)]^{-1}$ is SPR

$$I + KG(s)[I + K_1G(s)]^{-1} = [I + K_2G(s)][I + K_1G(s)]^{-1}$$

Theorem 7.8 (Circle Criterion)

The system is absolutely stable if

- $\psi \in [K_1, \infty]$ and $G(s)[I + K_1G(s)]^{-1}$ is SPR, or
- $\psi \in [K_1, K_2]$ and $[I + K_2G(s)][I + K_1G(s)]^{-1}$ is SPR

If the sector condition is satisfied only on a set $Y \subset R^m$, then the foregoing conditions ensure absolute stability with finite domain

Scalar Case: $\psi \in [\alpha, \beta],\ \beta > \alpha$

The system is absolutely stable if

$$\frac{1 + \beta G(s)}{1 + \alpha G(s)} \quad \text{is Hurwitz and}$$

$$\mathrm{Re}\left[\frac{1 + \beta G(j\omega)}{1 + \alpha G(j\omega)}\right] > 0, \quad \forall\ \omega \in [0, \infty]$$

Case 1: $\alpha > 0$

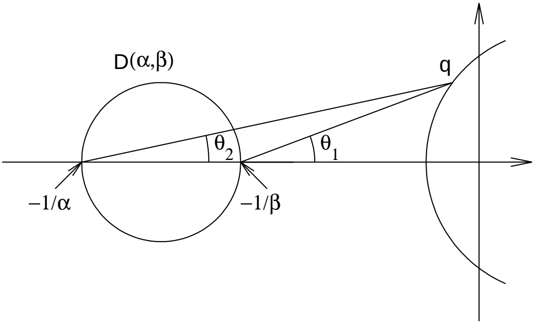
By the Nyquist criterion

$$\frac{1+\beta G(s)}{1+\alpha G(s)} = \frac{1}{1+\alpha G(s)} + \frac{\beta G(s)}{1+\alpha G(s)}$$

is Hurwitz if the Nyquist plot of $G(j\omega)$ does not intersect the point $-(1/\alpha)+j0$ and encircles it p times in the counterclockwise direction, where p is the number of poles of $G(s)$ in the open right-half complex plane

$$\frac{1+\beta G(j\omega)}{1+\alpha G(j\omega)} > 0 \quad \Leftrightarrow \quad \frac{\frac{1}{\beta}+G(j\omega)}{\frac{1}{\alpha}+G(j\omega)} > 0$$

$$\mathrm{Re}\left[\frac{\frac{1}{\beta}+G(j\omega)}{\frac{1}{\alpha}+G(j\omega)}\right] > 0, \quad \forall\ \omega \in [0,\infty]$$



The system is absolutely stable if the Nyquist plot of $G(j\omega)$ does not enter the disk $D(\alpha,\beta)$ and encircles it p times in the counterclockwise direction

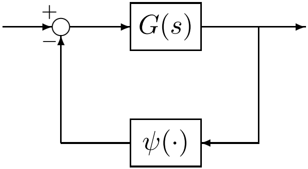
Theorem 7.9

Consider an SISO $G(s)$ and $\psi \in [\alpha, \beta]$. Then, the system is absolutely stable if one of the following conditions is satisfied.

1. $0 < \alpha < \beta$, the Nyquist plot of $G(s)$ does not enter the disk $D(\alpha, \beta)$ and encircles it p times in the counterclockwise direction, where p is the number of poles of $G(s)$ with positive real parts
2. $0 = \alpha < \beta$, $G(s)$ is Hurwitz and the Nyquist plot of $G(s)$ lies to the right of the vertical line $\mathrm{Re}[s] = -1/\beta$.
3. $\alpha < 0 < \beta$, $G(s)$ is Hurwitz and the Nyquist plot of $G(s)$ lies in the interior of the disk $D(\alpha, \beta)$.

If the sector condition is satisfied only on an interval $[a, b]$, then the foregoing conditions ensure absolute stability with finite domain

Popov Criterion



$$\dot{x} = Ax + Bu, \quad y = Cx$$

(A, B) controllable, (A, C) observable

$$u_i = -\psi_i(y_i), \quad \psi_i \in [0, k_i], \quad 1 \le i \le m, \quad (0 < k_i \le \infty)$$

$$G(s) = C(sI - A)^{-1}B$$

$$\Gamma = \mathrm{diag}(\gamma_1, \ldots, \gamma_m), \quad M = \mathrm{diag}(1/k_1, \cdots, 1/k_m)$$

Theorem 7.10

The system is absolutely stable if for $1 \le i \le m$,

$$\psi_i \in [0, k_i], \quad 0 < k_i \le \infty$$

and there is $\gamma_i \ge 0$, with $(1 + \lambda_k \gamma_i) \ne 0$ for every eigenvalue λ_k of A, such that

$$M + (I + s\Gamma)G(s) \quad \text{is SPR}$$

If the sector condition $\psi_i \in [0, k_i]$ is satisfied only on a set $Y \subset R^m$, then the system is absolutely stable with finite domain

Scalar case

$$\frac{1}{k} + (1 + s\gamma)G(s)$$

is SPR if $G(s)$ is Hurwitz and

$$\frac{1}{k} + \mathrm{Re}[G(j\omega)] - \gamma\omega \mathrm{Im}[G(j\omega)] > 0, \quad \forall\ \omega \in [0, \infty)$$

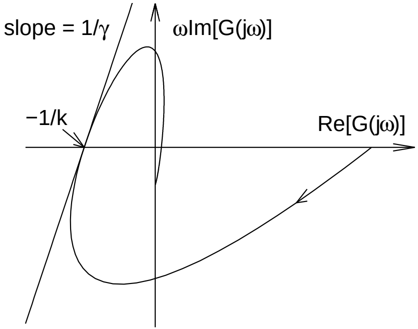
If

$$\lim_{\omega\to\infty} \left\{ \frac{1}{k} + \mathrm{Re}[G(j\omega)] - \gamma\omega \mathrm{Im}[G(j\omega)] \right\} = 0$$

we also need

$$\lim_{\omega\to\infty} \omega^2 \left\{ \frac{1}{k} + \mathrm{Re}[G(j\omega)] - \gamma\omega \mathrm{Im}[G(j\omega)] \right\} > 0$$

$$\frac{1}{k} + \mathrm{Re}[G(j\omega)] - \gamma\omega\mathrm{Im}[G(j\omega)] > 0, \quad \forall\ \omega \in [0, \infty)$$



Popov Plot

Example

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_2 - h(y), \quad y = x_1$$

$$\dot{x}_2 = -\alpha x_1 - x_2 - h(y) + \alpha x_1, \quad \alpha > 0$$

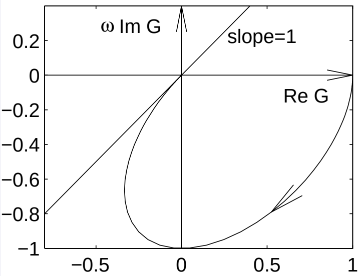
$$G(s) = \frac{1}{s^2 + s + \alpha}, \qquad \psi(y) = h(y) - \alpha y$$

$$h \in [\alpha, \beta] \;\; \Rightarrow \;\; \psi \in [0, k] \qquad\quad (k = \beta - \alpha > 0)$$

$$\gamma > 1 \;\; \Rightarrow \;\; \frac{\alpha - \omega^2 + \gamma\omega^2}{(\alpha - \omega^2)^2 + \omega^2} > 0, \;\; \forall\, \omega \in [0, \infty)$$

$$\text{and} \quad \lim_{\omega \to \infty} \frac{\omega^2(\alpha - \omega^2 + \gamma\omega^2)}{(\alpha - \omega^2)^2 + \omega^2} = \gamma - 1 > 0$$

The system is absolutely stable for $\psi \in [0, \infty]$ $(h \in [\alpha, \infty])$



Compare with the circle criterion $(\gamma = 0)$

$$\frac{1}{k} + \frac{\alpha - \omega^2}{(\alpha - \omega^2)^2 + \omega^2} > 0, \quad \forall\, \omega \in [0, \infty], \quad \text{for } k < 1 + 2\sqrt{\alpha}$$