Nonlinear Control

Lecture # 8

Special nonlinear Forms

Normal Form

Relative Degree

$$\dot{x} = f(x) + g(x)u, \qquad y = h(x)$$

where f, g, and h are sufficiently smooth in a domain D
$f : D \to R^n$ and $g : D \to R^n$ are called vector fields on D

$$\dot{y} = \frac{\partial h}{\partial x}[f(x) + g(x)u] \stackrel{\text{def}}{=} L_f h(x) + L_g h(x)\ u$$

$$L_f h(x) = \frac{\partial h}{\partial x} f(x)$$

is the *Lie Derivative* of h with respect to f or along f

$$L_g L_f h(x) = \frac{\partial (L_f h)}{\partial x} g(x)$$

$$L_f^2 h(x) = L_f L_f h(x) = \frac{\partial (L_f h)}{\partial x} f(x)$$

$$L_f^k h(x) = L_f L_f^{k-1} h(x) = \frac{\partial (L_f^{k-1} h)}{\partial x} f(x)$$

$$L_f^0 h(x) = h(x)$$

$$\dot{y} = L_f h(x) + L_g h(x)\ u$$

$$L_g h(x) = 0 \quad \Rightarrow \quad \dot{y} = L_f h(x)$$

$$y^{(2)} = \frac{\partial (L_f h)}{\partial x} [f(x) + g(x)u] = L_f^2 h(x) + L_g L_f h(x)\ u$$

$$L_g L_f h(x) = 0 \quad \Rightarrow \quad y^{(2)} = L_f^2 h(x)$$

$$y^{(3)} = L_f^3 h(x) + L_g L_f^2 h(x)\ u$$

$$L_g L_f^{i-1} h(x) = 0, \quad i = 1, 2, \ldots, \rho - 1; \qquad L_g L_f^{\rho-1} h(x) \neq 0$$

$$y^{(\rho)} = L_f^\rho h(x) + L_g L_f^{\rho-1} h(x)\ u$$

Definition 8.1

The system

$$\dot{x} = f(x) + g(x)u, \qquad y = h(x)$$

has relative degree ρ, $1 \leq \rho \leq n$, in $\mathcal{R} \subset D$ if $\forall\ x \in \mathcal{R}$

$$L_g L_f^{i-1} h(x) = 0, \quad i = 1, 2, \ldots, \rho - 1; \qquad L_g L_f^{\rho-1} h(x) \neq 0$$

Example 8.1 (Read)

Controlled van der Pol equation

$$\dot{x}_1 = x_2/\varepsilon, \quad \dot{x}_2 = \varepsilon[-x_1 + x_2 - \tfrac{1}{3}x_2^3 + u], \quad y = x_1$$

$$\dot{y} = \dot{x}_1 = x_2/\varepsilon, \quad \ddot{y} = \dot{x}_2/\varepsilon = -x_1 + x_2 - \tfrac{1}{3}x_2^3 + u$$

Relative degree two over R^2

$$\dot{x}_1 = x_2/\varepsilon, \quad \dot{x}_2 = \varepsilon[-x_1 + x_2 - \tfrac{1}{3}x_2^3 + u], \quad y = x_2$$

$\dot{y} = \varepsilon[-x_1 + x_2 - \tfrac{1}{3}x_2^3 + u]$, Relative degree one over R^2

$$\dot{x}_1 = x_2/\varepsilon, \quad \dot{x}_2 = \varepsilon[-x_1 + x_2 - \tfrac{1}{3}x_2^3 + u], \quad y = \tfrac{1}{2}(\varepsilon^2 x_1^2 + x_2^2)$$

$$\dot{y} = \varepsilon^2 x_1 \dot{x}_1 + x_2 \dot{x}_2 = \varepsilon x_2^2 - (\varepsilon/3)x_2^4 + \varepsilon x_2 u$$

Relative degree one in $\{x_2 \neq 0\}$

Example 8.2 (Field-controlled DC motor)

$$\begin{aligned} \dot{x}_1 &= d_1(-x_1 - x_2x_3 + V_a) \\ \dot{x}_2 &= d_2[-f_e(x_2) + u] \\ \dot{x}_3 &= d_3(x_1x_2 - bx_3) \\ y &= x_3 \end{aligned}$$

$$\dot{y} = \dot{x}_3 = d_3(x_1x_2 - bx_3)$$

$$\ddot{y} = d_3(x_1\dot{x}_2 + \dot{x}_1x_2 - b\dot{x}_3) = (\cdots) + d_2d_3x_1u$$

Relative degree one in $\{x_1 \neq 0\}$

Example 8.3

$$H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_0}$$

$$\dot{x} = Ax + Bu, \qquad y = Cx$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & & & \dots & 0 \\ 0 & 0 & 1 & \dots & & & \dots & 0 \\ \vdots & & \ddots & & & & & \vdots \\ & & & \ddots & & & & \\ & & & & \ddots & & & \vdots \\ \vdots & & & & & \ddots & & 0 \\ 0 & & & & & & 0 & 1 \\ -a_0 & -a_1 & \dots & \dots & -a_m & \dots & \dots & -a_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \\ \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} b_0 & b_1 & \dots & \dots & b_m & 0 & \dots & 0 \end{bmatrix}$$

Change of variables:

$$z = T(x) = \left[\begin{array}{c} \phi_1(x) \\ \vdots \\ \phi_{n-\rho}(x) \\ ---\\ h(x) \\ \vdots \\ L_f^{\rho-1}h(x) \end{array}\right] \stackrel{\text{def}}{=} \left[\begin{array}{c} \phi(x) \\ --- \\ \psi(x) \end{array}\right] \stackrel{\text{def}}{=} \left[\begin{array}{c} \eta \\ --- \\ \xi \end{array}\right]$$

ϕ_1 to $\phi_{n-\rho}$ are chosen such that $T(x)$ is a diffeomorphism on a domain $D_x \subset \mathcal{R}$

$$\text{When } \rho = n, \quad z = T(x) = \psi(x) = \xi$$

$$
\begin{aligned}
\dot{\eta} &= \frac{\partial \phi}{\partial x}[f(x) + g(x)u] \;=\; f_0(\eta, \xi) + g_0(\eta, \xi)u \\
\dot{\xi}_i &= \xi_{i+1}, \quad 1 \le i \le \rho - 1 \\
\dot{\xi}_\rho &= L_f^\rho h(x) + L_g L_f^{\rho-1} h(x)\; u \\
y &= \xi_1
\end{aligned}
$$

Choose $\phi(x)$ such that $T(x)$ is a diffeomorphism and

$$
\frac{\partial \phi_i}{\partial x} g(x) = 0, \quad \text{for } 1 \le i \le n - \rho, \; \forall \; x \in D_x
$$

Always possible (at least locally)

$$
\dot{\eta} = f_0(\eta, \xi)
$$

Theorem 8.1

Suppose the system

$$\dot{x} = f(x) + g(x)u, \qquad y = h(x)$$

has relative degree ρ ($\leq n$) in $\mathcal{R}$. If $\rho = n$, then for every $x_0 \in \mathcal{R}$, a neighborhood N of x_0 exists such that the map $T(x) = \psi(x)$, restricted to N, is a diffeomorphism on N. If $\rho < n$, then, for every $x_0 \in \mathcal{R}$, a neighborhood N of x_0 and smooth functions $\phi_1(x), \ldots, \phi_{n-\rho}(x)$ exist such that

$$\frac{\partial \phi_i}{\partial x} g(x) = 0, \quad \text{for } 1 \leq i \leq n - \rho$$

is satisfied for all $x \in N$ and the map $T(x) = \left[\begin{array}{c} \phi(x) \\ \psi(x) \end{array} \right]$, restricted to N, is a diffeomorphism on N

Normal Form:

$$\begin{aligned}\dot{\eta} &= f_0(\eta, \xi)\\ \dot{\xi}_i &= \xi_{i+1}, \quad 1 \le i \le \rho - 1\\ \dot{\xi}_\rho &= L_f^\rho h(x) + L_g L_f^{\rho-1} h(x)\ u\\ y &= \xi_1\end{aligned}$$

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \dots & 0\\ 0 & 0 & 1 & \dots & 0\\ \vdots & & \ddots & & \vdots\\ \vdots & & & 0 & 1\\ 0 & \dots & \dots & 0 & 0 \end{bmatrix}, \quad B_c = \begin{bmatrix} 0\\ 0\\ \vdots\\ 0\\ 1 \end{bmatrix}$$

$$C_c = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \end{bmatrix}$$

$$
\begin{aligned}
\dot{\eta} &= f_0(\eta, \xi) \\
\dot{\xi} &= A_c\xi + B_c\left[L_f^{\rho} h(x) + L_g L_f^{\rho-1} h(x)\ u\right] \\
y &= C_c\xi
\end{aligned}
$$

$$
\tilde{\psi}(\eta, \xi) = \left. L_f^{\rho} h(x)\right|_{x=T^{-1}(z)}, \tilde{\gamma}(\eta, \xi) = \left. L_g L_f^{\rho-1} h(x)\right|_{x=T^{-1}(z)}
$$

$$
\dot{\xi} = A_c\xi + B_c[\tilde{\psi}(\eta, \xi) + \tilde{\gamma}(\eta, \xi)u]
$$

If x^* is an open-loop equilibrium point at which $y = 0$; i.e., $f(x^*) = 0$ and $h(x^*) = 0$, then $\psi(x^*) = 0$. Take $\phi(x^*) = 0$ so that $z = 0$ is an open-loop equilibrium point.

Zero Dynamics

$$\begin{aligned}\dot{\eta} &= f_0(\eta, \xi)\\ \dot{\xi} &= A_c\xi + B_c\left[L_f^{\rho}h(x) + L_gL_f^{\rho-1}h(x)\ u\right]\\ y &= C_c\xi\end{aligned}$$

$$y(t) \equiv 0 \ \Rightarrow\ \xi(t) \equiv 0 \ \Rightarrow\ u(t) \equiv -\frac{L_f^{\rho}h(x(t))}{L_gL_f^{\rho-1}h(x(t))}$$

$$\Rightarrow\ \dot{\eta} = f_0(\eta, 0)$$

Definition

The equation $\dot{\eta} = f_0(\eta, 0)$ is called the *zero dynamics* of the system. The system is said to be *minimum phase* if the zero dynamics have an asymptotically stable equilibrium point in the domain of interest (at the origin if $T(0) = 0$)

$$Z^* = \{x \in \mathcal{R} \mid h(x) = L_f h(x) = \cdots = L_f^{\rho-1} h(x) = 0\}$$

$$y(t) \equiv 0 \;\Rightarrow\; x(t) \in Z^*$$

$$\Rightarrow\; u = u^*(x) \stackrel{\text{def}}{=} -\left.\frac{L_f^\rho h(x)}{L_g L_f^{\rho-1} h(x)}\right|_{x \in Z^*}$$

The restricted motion of the system is described by

$$\dot{x} = f^*(x) \stackrel{\text{def}}{=} \left[f(x) - g(x)\frac{L_f^\rho h(x)}{L_g L_f^{\rho-1} h(x)}\right]_{x \in Z^*}$$

Example 8.4

$$\dot{x}_1 = x_2/\varepsilon, \quad \dot{x}_2 = \varepsilon[-x_1 + x_2 - \tfrac{1}{3}x_2^3 + u], \quad y = x_2$$

$$\dot{y} = \dot{x}_2 = \varepsilon[-x_1 + x_2 - \tfrac{1}{3}x_2^3 + u] \;\Rightarrow\; \rho = 1$$

The system is in the normal form with $\eta = x_1$ and $\xi = x_2$

$$y(t) \equiv 0 \;\Rightarrow x_2(t) \equiv 0 \;\Rightarrow\; \dot{x}_1 = 0$$

Non-minimum phase

Example 8.5

$$\dot{x}_1 = -x_1 + \frac{2 + x_3^2}{1 + x_3^2}\, u,\ \ \dot{x}_2 = x_3,\ \ \dot{x}_3 = x_1 x_3 + u,\ \ y = x_2$$

$$\dot{y} = \dot{x}_2 = x_3$$

$$\ddot{y} = \dot{x}_3 = x_1 x_3 + u \quad \Rightarrow \quad \rho = 2$$

$$Z^* = \{x_2 = x_3 = 0\}$$

$$u = u^*(x) = 0 \quad \Rightarrow \quad \dot{x}_1 = -x_1$$

Minimum phase

Find $\phi(x)$ such that

$$\phi(0) = 0, \quad \frac{\partial \phi}{\partial x} g(x) = \begin{bmatrix} \frac{\partial \phi}{\partial x_1}, & \frac{\partial \phi}{\partial x_2}, & \frac{\partial \phi}{\partial x_3} \end{bmatrix} \begin{bmatrix} \frac{2+x_3^2}{1+x_3^2} \\ 0 \\ 1 \end{bmatrix} = 0$$

and

$$T(x) = \begin{bmatrix} \phi(x) & x_2 & x_3 \end{bmatrix}^T$$

is a diffeomorphism

$$\frac{\partial \phi}{\partial x_1} \cdot \frac{2 + x_3^2}{1 + x_3^2} + \frac{\partial \phi}{\partial x_3} = 0$$

$$\phi(x) = x_1 - x_3 - \tan^{-1} x_3$$

$$T(x) = \begin{bmatrix} x_1 - x_3 - \tan^{-1} x_3 \\ x_2 \\ x_3 \end{bmatrix}, \quad \frac{\partial T}{\partial x} = \begin{bmatrix} 1 & 0 & \star \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$T(x)$ is a global diffeomorphism

$$\begin{aligned} \dot{\eta} &= -\left(\eta + \xi_2 + \tan^{-1} \xi_2\right)\left(1 + \frac{2 + \xi_2^2}{1 + \xi_2^2}\xi_2\right) \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \left(\eta + \xi_2 + \tan^{-1} \xi_2\right)\xi_2 + u \\ y &= \xi_1 \end{aligned}$$

Controller Form

Definition

A nonlinear system is in the controller form if

$$\dot{x} = Ax + B[\psi(x) + \gamma(x)u]$$

where (A, B) is controllable and $\gamma(x)$ is a nonsingular matrix for all x in the domain of interest

$$u = \gamma^{-1}(x)[-\psi(x) + v] \quad \Rightarrow \quad \dot{x} = Ax + Bv$$

Any system that can be represented in the controller form is said to be feedback linearizable

Example 8.7 (m-link robot)

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + D\dot{q} + g(q) = u$$

q is an m-dimensional vector of joint positions and $M(q)$ is a nonsingular inertial matrix

$$x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}, \quad A = \begin{bmatrix} 0 & I_m \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ I_m \end{bmatrix}$$

$$\psi = -M^{-1}(C\dot{q} + D\dot{q} + g), \qquad \gamma = M^{-1}$$

An n-dimensional single-input system

$$\dot{x} = f(x) + g(x)u$$

is transformable into the controller form if and only if $\exists\ h(x)$ such that

$$\dot{x} = f(x) + g(x)u, \qquad y = h(x)$$

has relative degree n

Search for a smooth function $h(x)$ such that

$$L_g L_f^{i-1} h(x) = 0, \quad i = 1, 2, \ldots, n-1, \quad \text{and} \quad L_g L_f^{n-1} h(x) \neq 0$$

$$T(x) = \mathrm{col}\left(\ h(x), \quad L_f h(x), \quad \cdots \quad L_f^{n-1} h(x) \ \right)$$

The Lie Bracket: For two vector fields f and g, the *Lie bracket* $[f, g]$ is a third vector field defined by

$$[f, g](x) = \frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x)$$

Notation:

$$ad_f^0 g(x) = g(x), \qquad ad_f g(x) = [f, g](x)$$

$$ad_f^k g(x) = [f, ad_f^{k-1} g](x), \quad k \geq 1$$

Properties:

- $[f, g] = -[g, f]$
- For constant vector fields f and g, $[f, g] = 0$

Example 8.8

$$f = \begin{bmatrix} x_2 \\ -\sin x_1 - x_2 \end{bmatrix}, \quad g = \begin{bmatrix} 0 \\ x_1 \end{bmatrix}$$

$$[f, g] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ -\sin x_1 - x_2 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\cos x_1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ x_1 \end{bmatrix}$$

$$ad_f g = [f, g] = \begin{bmatrix} -x_1 \\ x_1 + x_2 \end{bmatrix}$$

$$f = \begin{bmatrix} x_2 \\ -\sin x_1 - x_2 \end{bmatrix}, \quad ad_f g = \begin{bmatrix} -x_1 \\ x_1 + x_2 \end{bmatrix}$$

$$\begin{aligned} ad_f^2 g &= [f, ad_f g] \\ &= \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ -\sin x_1 - x_2 \end{bmatrix} \\ &\quad - \begin{bmatrix} 0 & 1 \\ -\cos x_1 & -1 \end{bmatrix} \begin{bmatrix} -x_1 \\ x_1 + x_2 \end{bmatrix} \\ &= \begin{bmatrix} -x_1 - 2x_2 \\ x_1 + x_2 - \sin x_1 - x_1 \cos x_1 \end{bmatrix} \end{aligned}$$

Example 8.9

$f(x) = Ax$, g is a constant vector field

$$ad_f g = [f, g] = -Ag, \;\; ad_f^2 g = [f, ad_f g] = -A(-Ag) = A^2 g$$

$$ad_f^k g = (-1)^k A^k g$$

Distribution: For vector fields f_1, f_2, ..., f_k on $D \subset R^n$, let

$$\Delta(x) = \text{span}\{f_1(x), f_2(x), \ldots, f_k(x)\}$$

The collection of all vector spaces $\Delta(x)$ for $x \in D$ is called a *distribution* and referred to by

$$\Delta = \text{span}\{f_1, f_2, \ldots, f_k\}$$

If $\dim(\Delta(x)) = k$ for all $x \in D$, we say that Δ is a nonsingular distribution on D, generated by $f_1, \ldots, f_k$

A distribution Δ is *involutive* if

$$g_1 \in \Delta \ \text{ and } \ g_2 \in \Delta \ \Rightarrow \ [g_1, g_2] \in \Delta$$

If Δ is a nonsingular distribution, generated by $f_1, \ldots, f_k$, then it is involutive if and only if

$$[f_i, f_j] \in \Delta, \ \ \forall \ 1 \leq i, j \leq k$$

Example 8.10

$D = R^3$; $\Delta = \text{span}\{f_1, f_2\}$

$$f_1 = \begin{bmatrix} 2x_2 \\ 1 \\ 0 \end{bmatrix}, \quad f_2 = \begin{bmatrix} 1 \\ 0 \\ x_2 \end{bmatrix}, \qquad \dim(\Delta(x)) = 2, \quad \forall\ x \in D$$

$$[f_1, f_2] = \frac{\partial f_2}{\partial x} f_1 - \frac{\partial f_1}{\partial x} f_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{rank}\ [f_1(x), f_2(x), [f_1, f_2](x)] =$$
$$\text{rank} \begin{bmatrix} 2x_2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & x_2 & 1 \end{bmatrix} = 3, \quad \forall\ x \in D$$

Δ is not involutive

Example 8.11

$$D = \{x \in R^3 \mid x_1^2 + x_3^2 \neq 0\}; \; \Delta = \mathrm{span}\{f_1, f_2\}$$

$$f_1 = \begin{bmatrix} 2x_3 \\ -1 \\ 0 \end{bmatrix}, \; f_2 = \begin{bmatrix} -x_1 \\ -2x_2 \\ x_3 \end{bmatrix}, \; \dim(\Delta(x)) = 2, \; \forall \; x \in D$$

$$[f_1, f_2] = \frac{\partial f_2}{\partial x} f_1 - \frac{\partial f_1}{\partial x} f_2 = \begin{bmatrix} -4x_3 \\ 2 \\ 0 \end{bmatrix}$$

$$\mathrm{rank} \begin{bmatrix} 2x_3 & -x_1 & -4x_3 \\ -1 & -2x_2 & 2 \\ 0 & x_3 & 0 \end{bmatrix} = 2, \;\; \forall \; x \in D$$

Δ is involutive

Theorem 8.2

The n-dimensional single-input system

$$\dot{x} = f(x) + g(x)u$$

is feedback linearizable in a neighborhood of $x_0 \in D$ if and only if there is a domain $D_x \subset D$, with $x_0 \in D_x$, such that

1. the matrix $\mathcal{G}(x) = [g(x), ad_f g(x), \ldots, ad_f^{n-1} g(x)]$ has rank n for all $x \in D_x$;
2. the distribution $\mathcal{D} = \text{span}\,\{g, ad_f g, \ldots, ad_f^{n-2} g\}$ is involutive in D_x.

Example 8.12 (Read)

$$\dot{x} = \begin{bmatrix} a \sin x_2 \\ -x_1^2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$ad_f g = [f, g] = -\frac{\partial f}{\partial x} g = \begin{bmatrix} -a \cos x_2 \\ 0 \end{bmatrix}$$

$$[g(x), ad_f g(x)] = \begin{bmatrix} 0 & -a \cos x_2 \\ 1 & 0 \end{bmatrix}$$

$\mathrm{rank}[g(x), ad_f g(x)] = 2, \ \forall \ x$ such that $\cos x_2 \neq 0$

$\mathrm{span}\{g\}$ is involutive

Find h such that $L_g h(x) = 0,$ and $L_g L_f h(x) \neq 0$

$$\frac{\partial h}{\partial x} g = \frac{\partial h}{\partial x_2} = 0 \quad \Rightarrow \quad h \text{ is independent of } x_2$$

$$L_f h(x) = \frac{\partial h}{\partial x_1} a \sin x_2$$

$$L_g L_f h(x) = \frac{\partial (L_f h)}{\partial x} g = \frac{\partial (L_f h)}{\partial x_2} = \frac{\partial h}{\partial x_1} a \cos x_2$$

$L_g L_f h(x) \neq 0$ in $D_0 = \{x \in R^2 | \cos x_2 \neq 0\}$ if $\frac{\partial h}{\partial x_1} \neq 0$

$$\text{Take } h(x) = x_1 \quad \Rightarrow \quad T(x) = \begin{bmatrix} h \\ L_f h \end{bmatrix} = \begin{bmatrix} x_1 \\ a \sin x_2 \end{bmatrix}$$

Example 8.13 (A single link manipulator with flexible joints)

$$f(x) = \begin{bmatrix} x_2 \\ -a\sin x_1 - b(x_1 - x_3) \\ x_4 \\ c(x_1 - x_3) \end{bmatrix}, \qquad g = \begin{bmatrix} 0 \\ 0 \\ 0 \\ d \end{bmatrix}$$

$$ad_f g = [f, g] = -\frac{\partial f}{\partial x} g = \begin{bmatrix} 0 \\ 0 \\ -d \\ 0 \end{bmatrix}$$

$$ad_f^2 g = [f, ad_f g] = -\frac{\partial f}{\partial x} ad_f g = \begin{bmatrix} 0 \\ bd \\ 0 \\ -cd \end{bmatrix}$$

$$ad_f^3 g = [f, ad_f^2 g] = -\frac{\partial f}{\partial x} ad_f^2 g = \begin{bmatrix} -bd \\ 0 \\ cd \\ 0 \end{bmatrix}$$

$$\text{rank} \begin{bmatrix} 0 & 0 & 0 & -bd \\ 0 & 0 & bd & 0 \\ 0 & -d & 0 & cd \\ d & 0 & -cd & 0 \end{bmatrix} = 4$$

$$\mathrm{span}(g, ad_f g, ad_f^2 g) \text{ is involutive} \quad \text{Why?}$$

The system is feedback linearizable. Find $h(x)$ such that

$$\frac{\partial(L_f^{i-1}h)}{\partial x}g = 0, \quad i = 1, 2, 3, \quad \frac{\partial(L_f^3 h)}{\partial x}g \neq 0, \quad h(0) = 0$$

$$\frac{\partial h}{\partial x}g = 0 \;\Rightarrow\; \frac{\partial h}{\partial x_4} = 0$$

$$L_f h(x) = \frac{\partial h}{\partial x_1}x_2 + \frac{\partial h}{\partial x_2}[-a\sin x_1 - b(x_1 - x_3)] + \frac{\partial h}{\partial x_3}x_4$$

$$\frac{\partial(L_f h)}{\partial x}g = 0 \;\Rightarrow\; \frac{\partial(L_f h)}{\partial x_4} = 0 \;\Rightarrow\; \frac{\partial h}{\partial x_3} = 0$$

$$L_f h(x) = \frac{\partial h}{\partial x_1} x_2 + \frac{\partial h}{\partial x_2}[-a \sin x_1 - b(x_1 - x_3)]$$

$$L_f^2 h(x) = \frac{\partial (L_f h)}{\partial x_1} x_2 + \frac{\partial (L_f h)}{\partial x_2}[-a \sin x_1 - b(x_1 - x_3)] + \frac{\partial (L_f h)}{\partial x_3} x_4$$

$$\frac{\partial (L_f^2 h)}{\partial x_4} = 0 \;\Rightarrow\; \frac{\partial (L_f h)}{\partial x_3} = 0 \;\Rightarrow\; \frac{\partial h}{\partial x_2} = 0$$

$$\frac{\partial (L_f^3 h)}{\partial x} g \neq 0 \;\Leftrightarrow\; \frac{\partial h}{\partial x_1} \neq 0$$

$$h(x) = x_1, \; T(x) = \begin{bmatrix} x_1 \\ x_2 \\ -a \sin x_1 - b(x_1 - x_3) \\ -a x_2 \cos x_1 - b(x_2 - x_4) \end{bmatrix}$$

Example 8.14 (Field-Controlled DC Motor)

$$f = \begin{bmatrix} d_1(-x_1 - x_2x_3 + V_a) \\ -d_2 f_e(x_2) \\ d_3(x_1x_2 - bx_3) \end{bmatrix}, \; g = \begin{bmatrix} 0 \\ d_2 \\ 0 \end{bmatrix}, \; f_e \in \mathcal{C}^2 \text{ for } x_2 \in J$$

$$ad_f g = \begin{bmatrix} d_1d_2x_3 \\ d_2^2 f_e'(x_2) \\ -d_2d_3x_1 \end{bmatrix}$$

$$ad_f^2 g = \begin{bmatrix} d_1d_2x_3(d_1 + d_2f_e'(x_2) - bd_3) \\ d_2^3(f_e'(x_2))^2 - d_2^3 f_2(x_2) f_e''(x_2) \\ d_1d_2d_3(x_1 - V_a) - d_2^2d_3x_1f_e'(x_2) - bd_2d_3^2x_1 \end{bmatrix}$$

$$\det \mathcal{G} = -2d_1^2 d_2^3 d_3 x_3 (x_1 - a)(1 - bd_3/d_1)$$

$$a = \tfrac{1}{2} V_a / (1 - bd_3/d_1) > 0$$

$$x_1 \neq a \text{ and } x_3 \neq 0 \;\Rightarrow\; \text{rank}(\mathcal{G}) = 3$$

$$[g, ad_f g] = d_2^2 f_e''(x_2) g \;\Rightarrow\; \text{span}\{g, ad_f g\} \text{ is involutive}$$

The conditions of Theorem 8.2 are satisfied in the domain

$$D_x = \{x_1 > a,\; x_2 \in J,\; x_3 > 0\}$$

Find $h(x)$ such that

$$\frac{\partial h}{\partial x} g = 0; \quad \frac{\partial (L_f h)}{\partial x} g = 0; \quad \frac{\partial (L_f^2 h)}{\partial x} g \neq 0$$

$$\frac{\partial h}{\partial x} g = 0 \;\Rightarrow\; \frac{\partial h}{\partial x_2} = 0$$

$$L_f h(x) = \frac{\partial h}{\partial x_1} d_1(-x_1 - x_2 x_3 + V_a) + \frac{\partial h}{\partial x_3} d_3(x_1 x_2 - b x_3)$$

$$\frac{\partial (L_f h)}{\partial x} g = 0 \;\Rightarrow\; \frac{\partial (L_f h)}{\partial x_2} = 0 \;\Rightarrow\; -d_1 x_3 \frac{\partial h}{\partial x_1} + d_3 x_1 \frac{\partial h}{\partial x_3} = 0$$

Take $h = d_3 x_1^2 + d_1 x_3^2 + c$

$$L_f h(x) = 2 d_1 d_3 x_1 (V_a - x_1) - 2 b d_1 d_3 x_3^2$$

$$L_f^2 h(x) = 2 d_1^2 d_3 (V_a - 2x_1)(-x_1 - x_2 x_3 + V_a) - 4 b d_1 d_3^2 x_3 (x_1 x_2 - b x_3)$$

$$\frac{\partial (L_f^2 h)}{\partial x} g = d_2 \frac{\partial (L_f^2 h)}{\partial x_2} = 4 d_1^2 d_2 d_3 (1 - b d_3/d_1) x_3 (x_1 - a) \neq 0$$