Nonlinear Control

Lecture # 9

State Feedback Stabilization

Basic Concepts

We want to stabilize the system

$$\dot{x} = f(x, u)$$

at the equilibrium point $x = x_{\text{ss}}$

Steady-State Problem: Find steady-state control u_{ss} s.t.

$$0 = f(x_{\text{ss}}, u_{\text{ss}})$$

$$x_\delta = x - x_{\text{ss}}, \qquad u_\delta = u - u_{\text{ss}}$$

$$\dot{x}_\delta = f(x_{\text{ss}} + x_\delta, u_{\text{ss}} + u_\delta) \stackrel{\text{def}}{=} f_\delta(x_\delta, u_\delta)$$

$$f_\delta(0, 0) = 0$$

$$u_\delta = \phi(x_\delta) \quad \Rightarrow \quad u = u_{\text{ss}} + \phi(x - x_{\text{ss}})$$

State Feedback Stabilization: Given

$$\dot{x} = f(x, u) \qquad\qquad [f(0, 0) = 0]$$

find

$$u = \phi(x) \qquad\qquad [\phi(0) = 0]$$

s.t. the origin is an asymptotically stable equilibrium point of

$$\dot{x} = f(x, \phi(x))$$

f and ϕ are locally Lipschitz functions

Notions of Stabilization

$$\dot{x} = f(x,u), \qquad u = \phi(x)$$

Local Stabilization: The origin of $\dot{x} = f(x,\phi(x))$ is asymptotically stable (e.g., linearization)

Regional Stabilization: The origin of $\dot{x} = f(x,\phi(x))$ is asymptotically stable and a given region G is a subset of the region of attraction (for all $x(0) \in G$, $\lim_{t\to\infty} x(t) = 0$) (e.g., $G \subset \Omega_c = \{V(x) \le c\}$ where Ω_c is an estimate of the region of attraction)

Global Stabilization: The origin of $\dot{x} = f(x,\phi(x))$ is globally asymptotically stable

Semiglobal Stabilization: The origin of $\dot{x} = f(x, \phi(x))$ is asymptotically stable and $\phi(x)$ can be designed such that any given compact set (no matter how large) can be included in the region of attraction (Typically $u = \phi_p(x)$ is dependent on a parameter p such that for any compact set G, p can be chosen to ensure that G is a subset of the region of attraction)

What is the difference between global stabilization and semiglobal stabilization?

Example 9.1

$$\dot{x} = x^2 + u$$

Linearization:

$$\dot{x} = u, \qquad u = -kx, \quad k > 0$$

Closed-loop system:

$$\dot{x} = -kx + x^2$$

Linearization of the closed-loop system yields $\dot{x} = -kx$. Thus, $u = -kx$ achieves local stabilization

The region of attraction is $\{x < k\}$. Thus, for any set $\{-a \le x \le b\}$ with $b < k$, the control $u = -kx$ achieves regional stabilization

The control $u = -kx$ does not achieve global stabilization

But it achieves semiglobal stabilization because any compact set $\{|x| \le r\}$ can be included in the region of attraction by choosing $k > r$

The control

$$u = -x^2 - kx$$

achieves global stabilization because it yields the linear closed-loop system $\dot{x} = -kx$ whose origin is globally exponentially stable

Linearization

$$\dot{x} = f(x, u)$$

$f(0,0) = 0$ and f is continuously differentiable in a domain $D_x \times D_u$ that contains the origin $(x = 0,\ u = 0)$ ($D_x \subset R^n$, $D_u \subset R^m$)

$$\dot{x} = Ax + Bu$$

$$A = \frac{\partial f}{\partial x}(x, u)\bigg|_{x=0,u=0}; \qquad B = \frac{\partial f}{\partial u}(x, u)\bigg|_{x=0,u=0}$$

Assume (A, B) is stabilizable. Design a matrix K such that $(A - BK)$ is Hurwitz

$$u = -Kx$$

Closed-loop system:

$$\dot{x} = f(x, -Kx)$$

Linearization:

$$\begin{aligned} \dot{x} &= \left[\frac{\partial f}{\partial x}(x, -Kx) + \frac{\partial f}{\partial u}(x, -Kx)\ (-K)\right]_{x=0} x \\ &= (A - BK)x \end{aligned}$$

Since $(A - BK)$ is Hurwitz, the origin is an exponentially stable equilibrium point of the closed-loop system

Feedback Linearization

Consider the nonlinear system

$$\dot{x} = f(x) + G(x)u$$

$$f(0) = 0, \quad x \in R^n, \quad u \in R^m$$

Suppose there is a change of variables $z = T(x)$, defined for all $x \in D \subset R^n$, that transforms the system into the controller form

$$\dot{z} = Az + B[\psi(x) + \gamma(x)u]$$

where (A, B) is controllable and $\gamma(x)$ is nonsingular for all $x \in D$

$$u = \gamma^{-1}(x)[-\psi(x) + v] \quad \Rightarrow \quad \dot{z} = Az + Bv$$

$$v = -Kz$$

Design K such that $(A - BK)$ is Hurwitz

The origin $z = 0$ of the closed-loop system

$$\dot{z} = (A - BK)z$$

is globally exponentially stable

$$u = \gamma^{-1}(x)[-\psi(x) - KT(x)]$$

Closed-loop system in the x-coordinates:

$$\dot{x} = f(x) + G(x)\gamma^{-1}(x)[-\psi(x) - KT(x)] \stackrel{\text{def}}{=} f_c(x)$$

What can we say about the stability of $x = 0$ as an equilibrium point of $\dot{x} = f_c(x)$?

$$z = T(x) \;\Rightarrow\; \frac{\partial T}{\partial x}(x) f_c(x) = (A - BK)T(x)$$

$$\frac{\partial f_c}{\partial x}(0) = J^{-1}(A - BK)J, \qquad J = \frac{\partial T}{\partial x}(0) \text{ (nonsingular)}$$

The origin of $\dot{x} = f_c(x)$ is exponentially stable

Is $x = 0$ globally asymptotically stable? In general No

It is globally asymptotically stable if $T(x)$ is a global diffeomorphism

What information do we need to implement the control

$$u = \gamma^{-1}(x)[-\psi(x) - KT(x)] \text{ ?}$$

What is the effect of uncertainty in ψ, γ, and T?

Let $\hat{\psi}(x)$, $\hat{\gamma}(x)$, and $\hat{T}(x)$ be nominal models of $\psi(x)$, $\gamma(x)$, and $T(x)$

$$u = \hat{\gamma}^{-1}(x)[-\hat{\psi}(x) - K\hat{T}(x)]$$

Closed-loop system:

$$\dot{z} = (A - BK)z + B\Delta(z)$$

$$\dot{z} = (A - BK)z + B\Delta(z) \qquad (*)$$

$$V(z) = z^T P z, \qquad P(A - BK) + (A - BK)^T P = -I$$

Lemma 9.1

Suppose (*) is defined in $D_z \subset R^n$

- If $\|\Delta(z)\| \leq k\|z\| \ \forall \ z \in D_z$, $k < 1/(2\|PB\|)$, then the origin of (*) is exponentially stable. It is globally exponentially stable if $D_z = R^n$
- If $\|\Delta(z)\| \leq k\|z\| + \delta \ \forall \ z \in D_z$ and $B_r \subset D_z$, then there exist positive constants c_1 and c_2 such that if $\delta < c_1 r$ and $z(0) \in \{z^T P z \leq \lambda_{\min}(P) r^2\}$, $\|z(t)\|$ will be ultimately bounded by δc_2. If $D_z = R^n$, $\|z(t)\|$ will be globally ultimately bounded by δc_2 for any $\delta > 0$

Example 9.4 (Pendulum Equation)

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\sin(x_1 + \delta_1) - bx_2 + cu$$

$$u = \left(\frac{1}{c}\right)\left[\sin(x_1 + \delta_1) - (k_1 x_1 + k_2 x_2)\right]$$

$$A - BK = \begin{bmatrix} 0 & 1 \\ -k_1 & -(k_2 + b) \end{bmatrix}$$

$$u = \left(\frac{1}{\hat{c}}\right)\left[\sin(x_1 + \delta_1) - (k_1 x_1 + k_2 x_2)\right]$$

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -k_1 x_1 - (k_2 + b)x_2 + \Delta(x)$$

$$\Delta(x) = \left(\frac{c - \hat{c}}{\hat{c}}\right)\left[\sin(x_1 + \delta_1) - (k_1 x_1 + k_2 x_2)\right]$$

$$|\Delta(x)| \leq k\|x\| + \delta, \quad \forall\ x$$

$$k = \left|\frac{c-\hat{c}}{\hat{c}}\right| \left(1 + \sqrt{k_1^2 + k_2^2}\right), \quad \delta = \left|\frac{c-\hat{c}}{\hat{c}}\right| |\sin \delta_1|$$

$$P(A - BK) + (A - BK)^T P = -I, \quad P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$$

$$k < \frac{1}{2\sqrt{p_{12}^2 + p_{22}^2}} \ \Rightarrow \ \text{GUB}$$

$$\sin \delta_1 = 0 \ \&\ k < \frac{1}{2\sqrt{p_{12}^2 + p_{22}^2}} \ \Rightarrow \ \text{GES}$$

Is feedback linearization a good idea?

Example 9.5

$$\dot{x} = ax - bx^3 + u, \qquad a, b > 0$$

$$u = -(k+a)x + bx^3, \; k > 0, \;\; \Rightarrow \;\; \dot{x} = -kx$$

$-bx^3$ is a damping term. Why cancel it?

$$u = -(k+a)x, \; k > 0, \;\; \Rightarrow \;\; \dot{x} = -kx - bx^3$$

Which design is better?

Example 9.6

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -h(x_1) + u$$

$$h(0) = 0 \text{ and } x_1 h(x_1) > 0, \ \forall\ x_1 \neq 0$$

Feedback Linearization:

$$u = h(x_1) - (k_1 x_1 + k_2 x_2)$$

With $y = x_2$, the system is passive with

$$V = \int_0^{x_1} h(z)\ dz + \tfrac{1}{2} x_2^2$$

$$\dot{V} = h(x_1)\dot{x}_1 + x_2 \dot{x}_2 = yu$$

The control

$$u = -\sigma(x_2), \qquad \sigma(0) = 0,\ x_2\sigma(x_2) > 0\ \forall\ x_2 \neq 0$$

creates a feedback connection of two passive systems with storage function V

$$\dot{V} = -x_2\sigma(x_2)$$

$$x_2(t) \equiv 0 \ \Rightarrow\ \dot{x}_2(t) \equiv 0 \ \Rightarrow\ h(x_1(t)) \equiv 0 \ \Rightarrow\ x_1(t) \equiv 0$$

Asymptotic stability of the origin follows from the invariance principle

Which design is better?

The control $u = -\sigma(x_2)$ has two advantages:

- It does not use a model of h
- The flexibility in choosing σ can be used to reduce $|u|$

However, $u = -\sigma(x_2)$ cannot arbitrarily assign the rate of decay of $x(t)$. Linearization of the closed-loop system at the origin yields the characteristic equation

$$s^2 + \sigma'(0)s + h'(0) = 0$$

One of the two roots cannot be moved to the left of $\mathrm{Re}[s] = -\sqrt{h'(0)}$

Partial Feedback Linearization

Consider the nonlinear system

$$\dot{x} = f(x) + G(x)u \qquad [f(0) = 0]$$

Suppose there is a change of variables

$$z = \begin{bmatrix} \eta \\ \xi \end{bmatrix} = T(x) = \begin{bmatrix} T_1(x) \\ T_2(x) \end{bmatrix}$$

defined for all $x \in D \subset R^n$, that transforms the system into

$$\dot{\eta} = f_0(\eta, \xi), \quad \dot{\xi} = A\xi + B[\psi(x) + \gamma(x)u]$$

(A, B) is controllable and $\gamma(x)$ is nonsingular for all $x \in D$

$$u = \gamma^{-1}(x)[-\psi(x) + v]$$

$$\dot{\eta} = f_0(\eta, \xi), \qquad \dot{\xi} = A\xi + Bv$$

$v = -K\xi$, where $(A - BK)$ is Hurwitz

Lemma 9.2

The origin of the cascade connection

$$\dot{\eta} = f_0(\eta, \xi), \quad \dot{\xi} = (A - BK)\xi$$

is asymptotically (exponentially) stable if the origin of $\dot{\eta} = f_0(\eta, 0)$ is asymptotically (exponentially) stable

Proof

With $b > 0$ sufficiently small,

$$V(\eta, \xi) = bV_1(\eta) + \sqrt{\xi^T P \xi}, \quad (asymptotic)$$

$$V(\eta, \xi) = bV_1(\eta) + \xi^T P \xi, \quad (exponential)$$

If the origin of $\dot{\eta} = f_0(\eta, 0)$ is globally asymptotically stable, will the origin of

$$\dot{\eta} = f_0(\eta, \xi), \qquad \dot{\xi} = (A - BK)\xi$$

be globally asymptotically stable?

In general No

Example 9.7

The origin of $\dot{\eta} = -\eta$ is globally exponentially stable, but

$$\dot{\eta} = -\eta + \eta^2 \xi, \qquad \dot{\xi} = -k\xi, \quad k > 0$$

has a finite region of attraction $\{\eta\xi < 1 + k\}$

Example 9.8

$$\dot{\eta} = -\ \tfrac{1}{2}(1+\xi_2)\eta^3, \qquad \dot{\xi}_1 = \xi_2, \qquad \dot{\xi}_2 = v$$

The origin of $\dot{\eta} = -\ \frac{1}{2}\eta^3$ is globally asymptotically stable

$$v = -k^2\xi_1 - 2k\xi_2 \stackrel{\text{def}}{=} -K\xi \quad \Rightarrow \quad A - BK = \begin{bmatrix} 0 & 1 \\ -k^2 & -2k \end{bmatrix}$$

The eigenvalues of $(A - BK)$ are $-k$ and $-k$

$$e^{(A-BK)t} = \begin{bmatrix} (1+kt)e^{-kt} & te^{-kt} \\ -k^2te^{-kt} & (1-kt)e^{-kt} \end{bmatrix}$$

Peaking Phenomenon:

$$\max_t\{k^2te^{-kt}\} = \frac{k}{e} \to \infty \text{ as } k \to \infty$$

$$\xi_1(0) = 1,\ \xi_2(0) = 0 \quad \Rightarrow \quad \xi_2(t) = -k^2te^{-kt}$$

$$\dot{\eta} = -\ \tfrac{1}{2}\left(1 - k^2te^{-kt}\right)\eta^3, \qquad \eta(0) = \eta_0$$

$$\eta^2(t) = \frac{\eta_0^2}{1 + \eta_0^2[t + (1+kt)e^{-kt} - 1]}$$

If $\eta_0^2 > 1$, the system will have a finite escape time if k is chosen large enough

Lemma 9.3

The origin of

$$\dot{\eta} = f_0(\eta, \xi), \qquad \dot{\xi} = (A - BK)\xi$$

is globally asymptotically stable if the system $\dot{\eta} = f_0(\eta, \xi)$ is input-to-state stable

Proof

Apply Lemma 4.6

Model uncertainty can be handled as in the case of feedback linearization

Backstepping

$$\begin{aligned}\dot{\eta} &= f_a(\eta) + g_a(\eta)\xi \\ \dot{\xi} &= f_b(\eta,\xi) + g_b(\eta,\xi)u, \quad g_b \neq 0, \quad \eta \in R^n, \ \xi, \ u \in R\end{aligned}$$

Stabilize the origin using state feedback

View ξ as "virtual" control input to the system

$$\dot{\eta} = f_a(\eta) + g_a(\eta)\xi$$

Suppose there is $\xi = \phi(\eta)$ that stabilizes the origin of

$$\dot{\eta} = f_a(\eta) + g_a(\eta)\phi(\eta)$$

$$\frac{\partial V_a}{\partial \eta}[f_a(\eta) + g_a(\eta)\phi(\eta)] \leq -W(\eta)$$

$$z = \xi - \phi(\eta)$$

$$\begin{aligned}\dot{\eta} &= [f_a(\eta) + g_a(\eta)\phi(\eta)] + g_a(\eta)z \\ \dot{z} &= F(\eta,\xi) + g_b(\eta,\xi)u\end{aligned}$$

$$V(\eta,\xi) = V_a(\eta) + \tfrac{1}{2}z^2 = V_a(\eta) + \tfrac{1}{2}[\xi - \phi(\eta)]^2$$

$$\begin{aligned}\dot{V} &= \frac{\partial V_a}{\partial \eta}[f_a(\eta) + g_a(\eta)\phi(\eta)] + \frac{\partial V_a}{\partial \eta}g_a(\eta)z \\ &\quad + zF(\eta,\xi) + zg_b(\eta,\xi)u \\ &\leq -W(\eta) + z\left[\frac{\partial V_a}{\partial \eta}g_a(\eta) + F(\eta,\xi) + g_b(\eta,\xi)u\right]\end{aligned}$$

$$\dot{V} \leq -W(\eta) + z\left[\frac{\partial V_a}{\partial \eta} g_a(\eta) + F(\eta, \xi) + g_b(\eta, \xi) u\right]$$

$$u = -\frac{1}{g_b(\eta, \xi)}\left[\frac{\partial V_a}{\partial \eta} g_a(\eta) + F(\eta, \xi) + kz\right], \quad k > 0$$

$$\dot{V} \leq -W(\eta) - kz^2$$

Example 9.9

$$\dot{x}_1 = x_1^2 - x_1^3 + x_2, \qquad \dot{x}_2 = u$$

$$\dot{x}_1 = x_1^2 - x_1^3 + x_2$$

$$x_2 = \phi(x_1) = -x_1^2 - x_1 \quad \Rightarrow \quad \dot{x}_1 = -x_1 - x_1^3$$

$$V_a(x_1) = \tfrac{1}{2}x_1^2 \quad \Rightarrow \quad \dot{V}_a = -x_1^2 - x_1^4, \qquad \forall\ x_1 \in R$$

$$z_2 = x_2 - \phi(x_1) = x_2 + x_1 + x_1^2$$

$$\begin{aligned} \dot{x}_1 &= -x_1 - x_1^3 + z_2 \\ \dot{z}_2 &= u + (1 + 2x_1)(-x_1 - x_1^3 + z_2) \end{aligned}$$

$$V(x) = \tfrac{1}{2}x_1^2 + \tfrac{1}{2}z_2^2$$

$$\begin{aligned} \dot{V} &= x_1(-x_1 - x_1^3 + z_2) \\ &\quad + z_2[u + (1 + 2x_1)(-x_1 - x_1^3 + z_2)] \end{aligned}$$

$$\begin{aligned} \dot{V} &= -x_1^2 - x_1^4 \\ &\quad + z_2[x_1 + (1 + 2x_1)(-x_1 - x_1^3 + z_2) + u] \end{aligned}$$

$$u = -x_1 - (1 + 2x_1)(-x_1 - x_1^3 + z_2) - z_2$$

$$\dot{V} = -x_1^2 - x_1^4 - z_2^2$$

The origin is globally asymptotically stable

Example 9.10

$$\dot{x}_1 = x_1^2 - x_1^3 + x_2, \qquad \dot{x}_2 = x_3, \qquad \dot{x}_3 = u$$

$$\dot{x}_1 = x_1^2 - x_1^3 + x_2, \qquad \dot{x}_2 = x_3$$

$$x_3 = -x_1 - (1 + 2x_1)(-x_1 - x_1^3 + z_2) - z_2 \stackrel{\text{def}}{=} \phi(x_1, x_2)$$

$$V_a(x) = \tfrac{1}{2}x_1^2 + \tfrac{1}{2}z_2^2, \qquad \dot{V}_a = -x_1^2 - x_1^4 - z_2^2$$

$$z_3 = x_3 - \phi(x_1, x_2)$$

$$\begin{aligned} \dot{x}_1 &= x_1^2 - x_1^3 + x_2, \qquad \dot{x}_2 = \phi(x_1, x_2) + z_3 \\ \dot{z}_3 &= u - \frac{\partial \phi}{\partial x_1}(x_1^2 - x_1^3 + x_2) - \frac{\partial \phi}{\partial x_2}(\phi + z_3) \end{aligned}$$

$$V = V_a + \tfrac{1}{2}z_3^2$$

$$\begin{aligned}\dot{V} &= \frac{\partial V_a}{\partial x_1}(x_1^2 - x_1^3 + x_2) + \frac{\partial V_a}{\partial x_2}(z_3 + \phi) \\ &\quad + z_3\left[u - \frac{\partial \phi}{\partial x_1}(x_1^2 - x_1^3 + x_2) - \frac{\partial \phi}{\partial x_2}(z_3 + \phi)\right]\end{aligned}$$

$$\begin{aligned}\dot{V} &= -x_1^2 - x_1^4 - (x_2 + x_1 + x_1^2)^2 \\ &\quad + z_3\left[\frac{\partial V_a}{\partial x_2} - \frac{\partial \phi}{\partial x_1}(x_1^2 - x_1^3 + x_2) - \frac{\partial \phi}{\partial x_2}(z_3 + \phi) + u\right]\end{aligned}$$

$$u = -\frac{\partial V_a}{\partial x_2} + \frac{\partial \phi}{\partial x_1}(x_1^2 - x_1^3 + x_2) + \frac{\partial \phi}{\partial x_2}(z_3 + \phi) - z_3$$

The origin is globally asymptotically stable

Strict-Feedback Form

$$
\begin{aligned}
\dot{x} &= f_0(x) + g_0(x)z_1 \\
\dot{z}_1 &= f_1(x, z_1) + g_1(x, z_1)z_2 \\
\dot{z}_2 &= f_2(x, z_1, z_2) + g_2(x, z_1, z_2)z_3 \\
&\vdots \\
\dot{z}_{k-1} &= f_{k-1}(x, z_1, \ldots, z_{k-1}) + g_{k-1}(x, z_1, \ldots, z_{k-1})z_k \\
\dot{z}_k &= f_k(x, z_1, \ldots, z_k) + g_k(x, z_1, \ldots, z_k)u
\end{aligned}
$$

$$
g_i(x, z_1, \ldots, z_i) \neq 0 \quad \text{for } 1 \leq i \leq k
$$