# Nonlinear Systems and Control
# Lecture # 15
# Positive Real Transfer Functions
# &
# Connection with Lyapunov Stability

Definition: A $p \times p$ proper rational transfer function matrix $G(s)$ is positive real if

- poles of all elements of $G(s)$ are in $Re[s] \leq 0$
- for all real $\omega$ for which $j\omega$ is not a pole of any element of $G(s)$, the matrix $G(j\omega) + G^T(-j\omega)$ is positive semidefinite
- any pure imaginary pole $j\omega$ of any element of $G(s)$ is a simple pole and the residue matrix $\lim_{s \to j\omega}(s - j\omega)G(s)$ is positive semidefinite Hermitian

$G(s)$ is called strictly positive real if $G(s - \varepsilon)$ is positive real for some $\varepsilon > 0$

## Scalar Case ($p = 1$):

$$G(j\omega) + G^T(-j\omega) = 2Re[G(j\omega)]$$

$Re[G(j\omega)]$ is an even function of $\omega$. The second condition of the definition reduces to

$$Re[G(j\omega)] \geq 0, \ \forall\, \omega \in [0, \infty)$$

which holds when the Nyquist plot of of $G(j\omega)$ lies in the closed right-half complex plane

This is true only if the relative degree of the transfer function is zero or one

Lemma: A $p \times p$ proper rational transfer function matrix $G(s)$ is strictly positive real if and only if

- $G(s)$ is Hurwitz
- $G(j\omega) + G^T(-j\omega) > 0, \; \forall \, \omega \in R$
- $G(\infty) + G^T(\infty) > 0$ or

$$\lim_{\omega \to \infty} \omega^{2(p-q)} \det[G(j\omega) + G^T(-j\omega)] > 0$$

where $q = \text{rank}[G(\infty) + G^T(\infty)]$

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- $G(\infty) > 0$ or

$$\lim_{\omega \to \infty} \omega^2 Re[G(j\omega)] > 0$$

## Example:

$$G(s) = \frac{1}{s}$$

has a simple pole at $s = 0$ whose residue is $1$

$$Re[G(j\omega)] = Re\left[\frac{1}{j\omega}\right] = 0, \quad \forall\, \omega \neq 0$$

Hence, $G$ is positive real. It is not strictly positive real since

$$\frac{1}{(s - \varepsilon)}$$

has a pole in $Re[s] > 0$ for any $\varepsilon > 0$

Example:

$$G(s) = \frac{1}{s+a}, \; a > 0, \quad \text{is Hurwitz}$$

$$Re[G(j\omega)] = \frac{a}{\omega^2 + a^2} > 0, \quad \forall\, \omega \in [0, \infty)$$

$$\lim_{\omega \to \infty} \omega^2 Re[G(j\omega)] = \lim_{\omega \to \infty} \frac{\omega^2 a}{\omega^2 + a^2} = a > 0 \;\; \Rightarrow \;\; \text{G is SPR}$$

Example:

$$G(s) = \frac{1}{s^2 + s + 1}, \quad Re[G(j\omega)] = \frac{1 - \omega^2}{(1 - \omega^2)^2 + \omega^2}$$

G is not PR

## Example:

$$G(s) = \begin{bmatrix} \frac{s+2}{s+1} & \frac{1}{s+2} \\ \\ \frac{-1}{s+2} & \frac{2}{s+1} \end{bmatrix} \text{ is Hurwitz}$$

$$G(j\omega) + G^T(-j\omega) = \begin{bmatrix} \frac{2(2+\omega^2)}{1+\omega^2} & \frac{-2j\omega}{4+\omega^2} \\ \\ \frac{2j\omega}{4+\omega^2} & \frac{4}{1+\omega^2} \end{bmatrix} > 0, \quad \forall\, \omega \in R$$

$$G(\infty) + G^T(\infty) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \quad q = 1$$

$$\lim_{\omega \to \infty} \omega^2 \det[G(j\omega) + G^T(-j\omega)] = 4 \quad \Rightarrow \quad G \text{ is SPR}$$

Positive Real Lemma: Let

$$\boldsymbol{G(s) = C(sI - A)^{-1}B + D}$$

where $(\boldsymbol{A, B})$ is controllable and $(\boldsymbol{A, C})$ is observable. $\boldsymbol{G(s)}$ is positive real if and only if there exist matrices $\boldsymbol{P = P^T > 0}$, $\boldsymbol{L}$, and $\boldsymbol{W}$ such that

$$\begin{aligned} \boldsymbol{PA + A^T P} &= \boldsymbol{-L^T L} \\ \boldsymbol{PB} &= \boldsymbol{C^T - L^T W} \\ \boldsymbol{W^T W} &= \boldsymbol{D + D^T} \end{aligned}$$

**Kalman–Yakubovich–Popov Lemma:** Let

$$G(s) = C(sI - A)^{-1}B + D$$

where $(A, B)$ is controllable and $(A, C)$ is observable. $G(s)$ is strictly positive real if and only if there exist matrices $P = P^T > 0$, $L$, and $W$, and a positive constant $\varepsilon$ such that

$$\begin{aligned} PA + A^T P &= -L^T L - \varepsilon P \\ PB &= C^T - L^T W \\ W^T W &= D + D^T \end{aligned}$$

**Lemma:** The linear time-invariant minimal realization

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

with

$$G(s) = C(sI - A)^{-1}B + D$$

is

- passive if $G(s)$ is positive real
- strictly passive if $G(s)$ is strictly positive real

**Proof:** Apply the PR and KYP Lemmas, respectively, and use $V(x) = \frac{1}{2}x^T P x$ as the storage function

$$
\begin{aligned}
& u^T y - \frac{\partial V}{\partial x}(Ax + Bu) \\
&\quad = \; u^T(Cx + Du) - x^T P(Ax + Bu) \\
&\quad = \; u^T Cx + \tfrac{1}{2}u^T(D + D^T)u \\
&\qquad\; - \tfrac{1}{2}x^T(PA + A^T P)x - x^T PBu \\
&\quad = \; u^T(B^T P + W^T L)x + \tfrac{1}{2}u^T W^T W u \\
&\qquad\; + \tfrac{1}{2}x^T L^T Lx + \tfrac{1}{2}\varepsilon x^T Px - x^T PBu \\
&\quad = \; \tfrac{1}{2}(Lx + Wu)^T(Lx + Wu) + \tfrac{1}{2}\varepsilon x^T Px \;\geq \tfrac{1}{2}\varepsilon x^T Px
\end{aligned}
$$

In the case of the PR Lemma, $\varepsilon = 0$, and we conclude that the system is passive; in the case of the KYP Lemma, $\varepsilon > 0$, and we conclude that the system is strictly passive

## Connection with Lyapunov Stability

Lemma: If the system

$$\dot{x} = f(x, u), \qquad y = h(x, u)$$

is passive with a positive definite storage function $V(x)$, then the origin of $\dot{x} = f(x, 0)$ is stable

Proof:

$$u^T y \geq \frac{\partial V}{\partial x} f(x, u) \;\; \Rightarrow \;\; \frac{\partial V}{\partial x} f(x, 0) \leq 0$$

Lemma: If the system

$$\dot{x} = f(x,u), \qquad y = h(x,u)$$

is strictly passive, then the origin of $\dot{x} = f(x,0)$ is asymptotically stable. Furthermore, if the storage function is radially unbounded, the origin will be globally asymptotically stable

Proof: The storage function $V(x)$ is positive definite

$$u^T y \geq \frac{\partial V}{\partial x} f(x,u) + \psi(x) \;\; \Rightarrow \;\; \frac{\partial V}{\partial x} f(x,0) \leq -\psi(x)$$

Why is $V(x)$ positive definite? Let $\phi(t;x)$ be the solution of $\dot{z} = f(z,0)$, $z(0) = x$

$$\dot{V} \leq -\psi(x)$$

$$V(\phi(\tau, x)) - V(x) \leq -\int_0^\tau \psi(\phi(t; x))\, dt, \quad \forall\, \tau \in [0, \delta]$$

$$V(\phi(\tau, x)) \geq 0 \;\Rightarrow\; V(x) \geq \int_0^\tau \psi(\phi(t; x))\, dt$$

$$V(\bar{x}) = 0 \;\Rightarrow\; \int_0^\tau \psi(\phi(t; \bar{x}))\, dt = 0, \;\forall\, \tau \in [0, \delta]$$

$$\Rightarrow\; \psi(\phi(t; \bar{x})) \equiv 0 \;\Rightarrow\; \phi(t; \bar{x}) \equiv 0 \;\Rightarrow\; \bar{x} = 0$$

**Definition:** The system

$$\dot{x} = f(x, u), \qquad y = h(x, u)$$

is zero-state observable if no solution of $\dot{x} = f(x, 0)$ can stay identically in $S = \{h(x, 0) = 0\}$, other than the zero solution $x(t) \equiv 0$

## Linear Systems

$$\dot{x} = Ax, \quad y = Cx$$

Observability of $(A, C)$ is equivalent to

$$y(t) = Ce^{At}x(0) \equiv 0 \ \Leftrightarrow \ x(0) = 0 \ \Leftrightarrow \ x(t) \equiv 0$$

Lemma: If the system

$$\dot{x} = f(x, u), \qquad y = h(x, u)$$

is output strictly passive and zero-state observable, then the origin of $\dot{x} = f(x, 0)$ is asymptotically stable. Furthermore, if the storage function is radially unbounded, the origin will be globally asymptotically stable

Proof: The storage function $V(x)$ is positive definite

$$u^T y \geq \frac{\partial V}{\partial x} f(x, u) + y^T \rho(y) \;\Rightarrow\; \frac{\partial V}{\partial x} f(x, 0) \leq -y^T \rho(y)$$

$$\dot{V}(x(t)) \equiv 0 \;\Rightarrow\; y(t) \equiv 0 \;\Rightarrow x(t) \equiv 0$$

Apply the invariance principle

## Example

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -ax_1^3 - kx_2 + u, \quad y = x_2, \quad a, k > 0$$

$$V(x) = \tfrac{1}{4}ax_1^4 + \tfrac{1}{2}x_2^2$$

$$\dot{V} = ax_1^3 x_2 + x_2(-ax_1^3 - kx_2 + u) = -ky^2 + yu$$

The system is output strictly passive

$$y(t) \equiv 0 \;\Leftrightarrow\; x_2(t) \equiv 0 \;\Rightarrow\; ax_1^3(t) \equiv 0 \;\Rightarrow\; x_1(t) \equiv 0$$

The system is zero-state observable. $V$ is radially unbounded. Hence, the origin of the unforced system is globally asymptotically stable