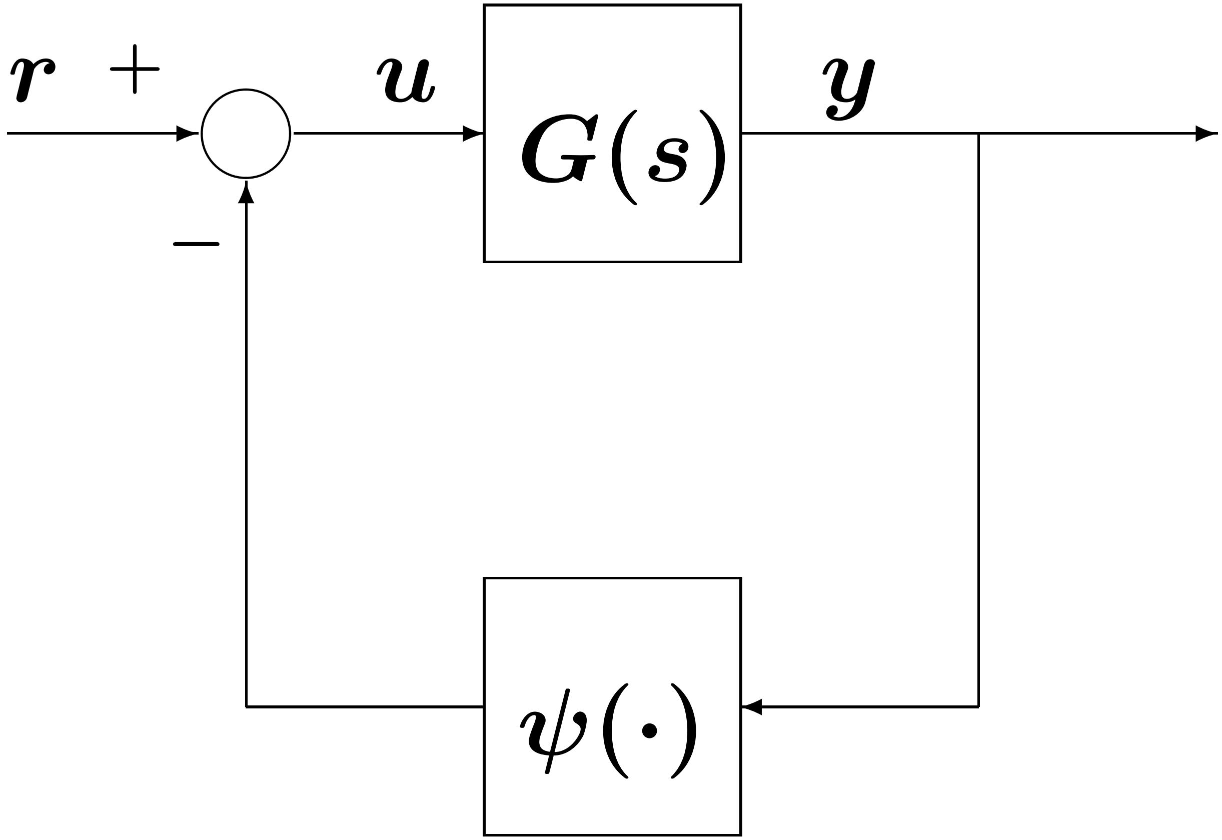
Nonlinear Systems and Control
Lecture # 17

Circle & Popov Criteria

Absolute Stability



The system is absolutely stable if (when $r = 0$) the origin is globally asymptotically stable for all memoryless time-invariant nonlinearities in a given sector

Circle Criterion

Suppose $G(s) = C(sI - A)^{-1}B + D$ is SPR and $\psi \in [0, \infty]$

$$
\begin{aligned}
\dot{x} &= Ax + Bu \\
y &= Cx + Du \\
u &= -\psi(y)
\end{aligned}
$$

By the KYP Lemma, $\exists\ P = P^T > 0,\ L, W, \varepsilon > 0$

$$
\begin{aligned}
PA + A^T P &= -L^T L - \varepsilon P \\
PB &= C^T - L^T W \\
W^T W &= D + D^T
\end{aligned}
$$

$$V(x) = \tfrac{1}{2} x^T P x$$

$$
\begin{aligned}
\dot{V} &= \tfrac{1}{2}x^T P\dot{x} + \tfrac{1}{2}\dot{x}^T Px \\
&= \tfrac{1}{2}x^T(PA + A^T P)x + x^T PBu \\
&= -\tfrac{1}{2}x^T L^T Lx - \tfrac{1}{2}\varepsilon x^T Px + x^T(C^T - L^T W)u \\
&= -\tfrac{1}{2}x^T L^T Lx - \tfrac{1}{2}\varepsilon x^T Px + (Cx + Du)^T u \\
&\quad - u^T Du - x^T L^T Wu
\end{aligned}
$$

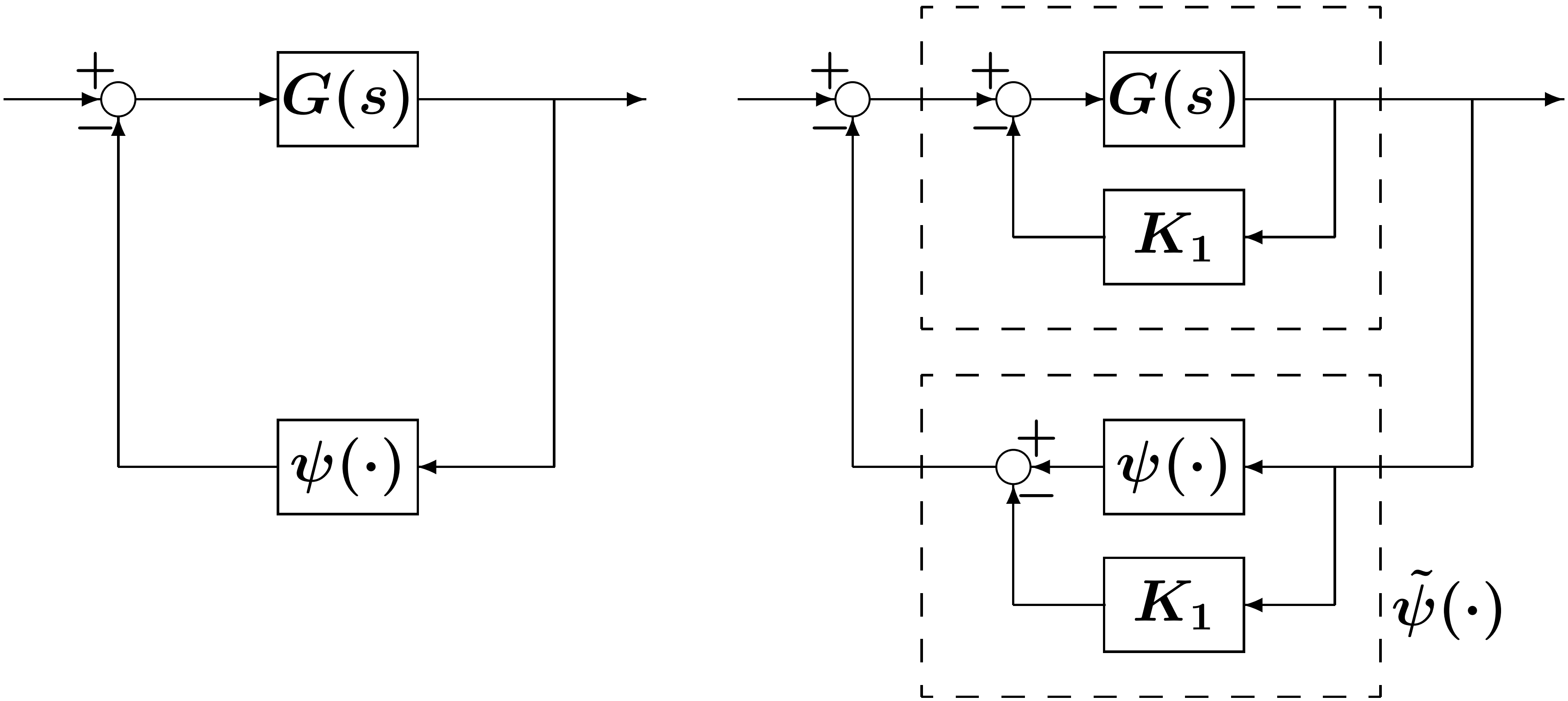
$$
u^T Du = \tfrac{1}{2}u^T(D + D^T)u = \tfrac{1}{2}u^T W^T Wu
$$

$$
\dot{V} = -\tfrac{1}{2}\varepsilon x^T Px - \tfrac{1}{2}(Lx + Wu)^T(Lx + Wu) - y^T\psi(y)
$$

$$
y^T\psi(y) \geq 0 \quad \Rightarrow \quad \dot{V} \leq -\tfrac{1}{2}\varepsilon x^T Px
$$

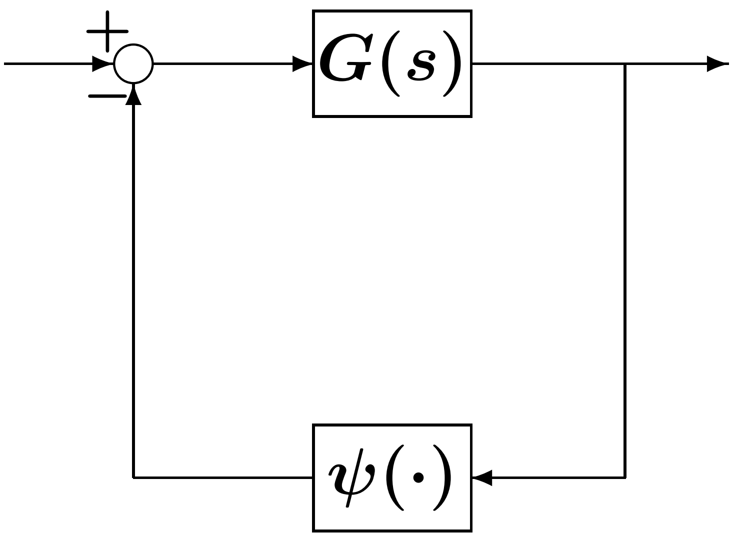
The origin is globally exponentially stable

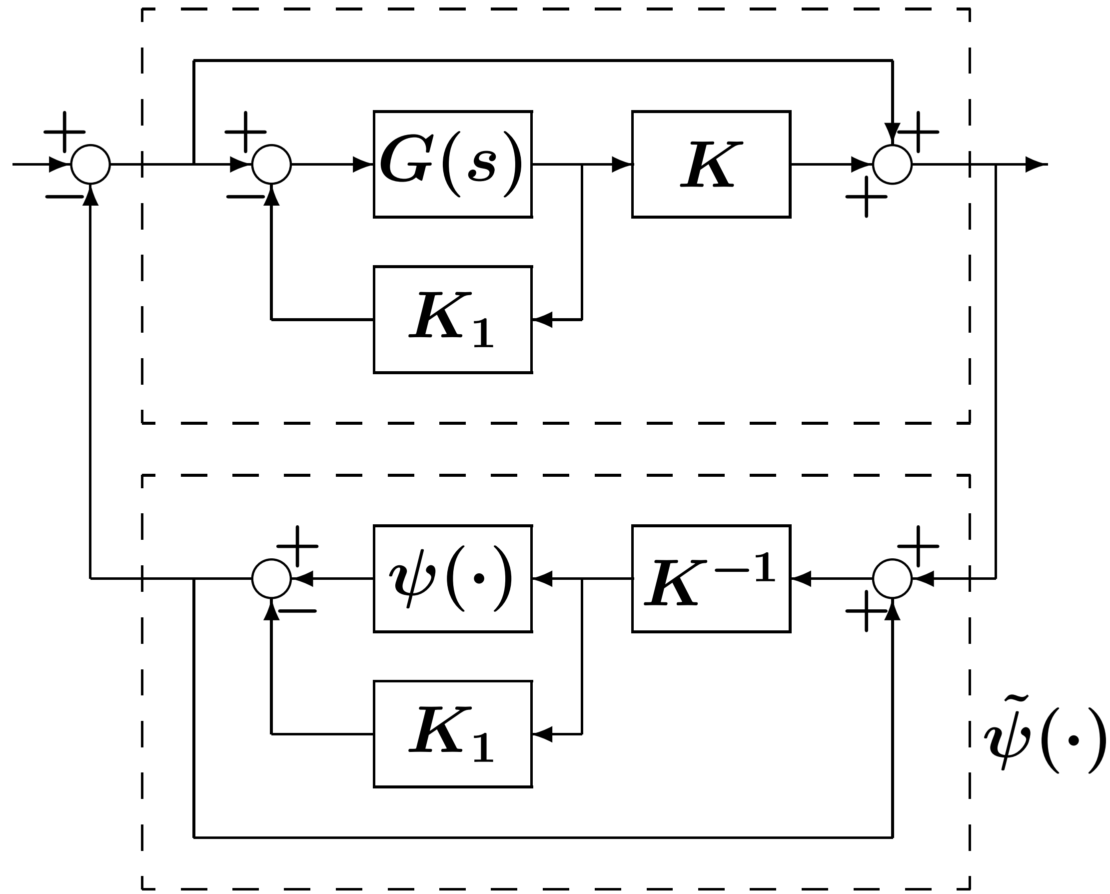
What if $\psi \in [K_1, \infty]$?



$\tilde{\psi} \in [0, \infty]$; hence the origin is globally exponentially stable if $G(s)[I + K_1 G(s)]^{-1}$ is SPR

What if $\psi \in [K_1, K_2]$?





$\tilde{\psi} \in [0, \infty]$; hence the origin is globally exponentially stable if $I + KG(s)[I + K_1 G(s)]^{-1}$ is SPR

$$I + KG(s)[I + K_1 G(s)]^{-1} = [I + K_2 G(s)][I + K_1 G(s)]^{-1}$$

Theorem (Circle Criterion): The system is absolutely stable if

- $\psi \in [K_1, \infty]$ and $G(s)[I + K_1 G(s)]^{-1}$ is SPR, or
- $\psi \in [K_1, K_2]$ and $[I + K_2 G(s)][I + K_1 G(s)]^{-1}$ is SPR

Scalar Case: $\psi \in [\alpha, \beta], \ \beta > \alpha$
The system is absolutely stable if

$$\frac{1 + \beta G(s)}{1 + \alpha G(s)} \quad \text{is Hurwitz and}$$

$$\mathrm{Re}\left[\frac{1 + \beta G(j\omega)}{1 + \alpha G(j\omega)}\right] > 0, \ \ \forall\, \omega \in [0, \infty]$$

Case 1: $\alpha > 0$

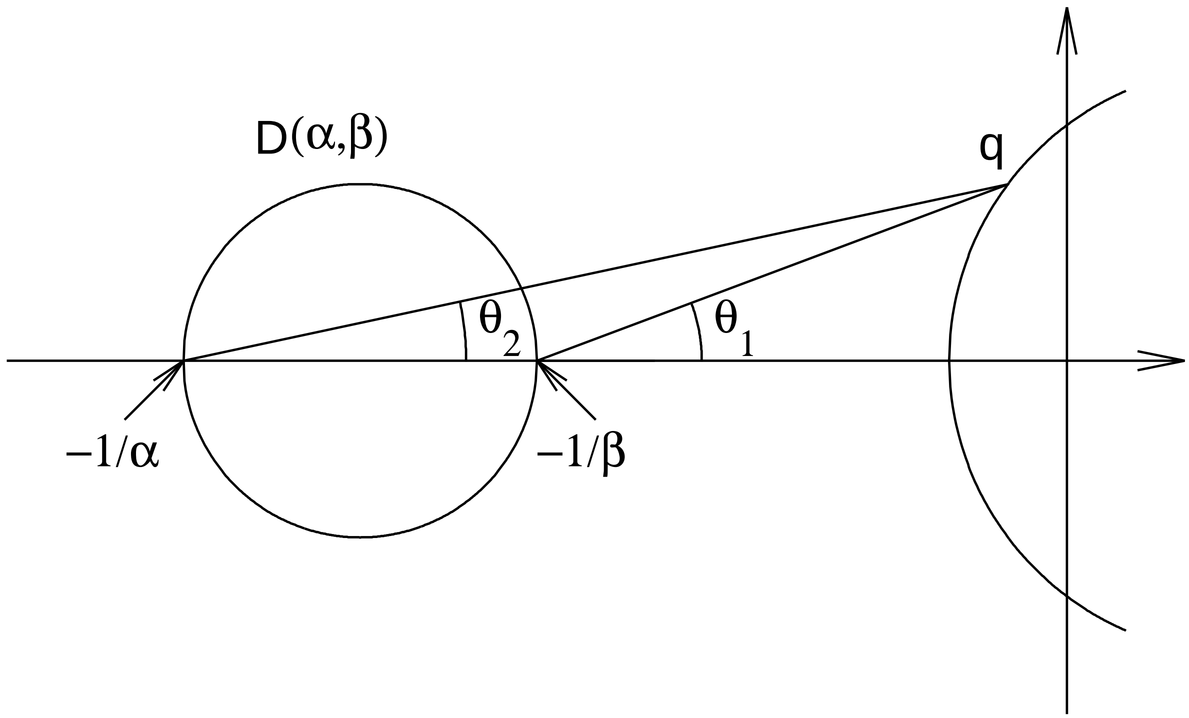
By the Nyquist criterion

$$\frac{1+\beta G(s)}{1+\alpha G(s)} = \frac{1}{1+\alpha G(s)} + \frac{\beta G(s)}{1+\alpha G(s)}$$

is Hurwitz if the Nyquist plot of $G(j\omega)$ does not intersect the point $-(1/\alpha) + j0$ and encircles it m times in the counterclockwise direction, where m is the number of poles of $G(s)$ in the open right-half complex plane

$$\frac{1+\beta G(j\omega)}{1+\alpha G(j\omega)} > 0 \quad \Leftrightarrow \quad \frac{\frac{1}{\beta} + G(j\omega)}{\frac{1}{\alpha} + G(j\omega)} > 0$$

$$\mathrm{Re}\left[\frac{\frac{1}{\beta}+G(j\omega)}{\frac{1}{\alpha}+G(j\omega)}\right] > 0, \quad \forall\, \omega \in [0, \infty]$$



The system is absolutely stable if the Nyquist plot of $G(j\omega)$ does not enter the disk $D(\alpha, \beta)$ and encircles it m times in the counterclockwise direction

Case 2: $\alpha = 0$

$$1 + \beta G(s)$$

$$\mathrm{Re}[1 + \beta G(j\omega)] > 0, \;\; \forall\, \omega \in [0, \infty]$$

$$\mathrm{Re}[G(j\omega)] > -\frac{1}{\beta}, \;\; \forall\, \omega \in [0, \infty]$$

The system is absolutely stable if $G(s)$ is Hurwitz and the Nyquist plot of $G(j\omega)$ lies to the right of the vertical line defined by $\mathrm{Re}[s] = -1/\beta$

Case 3: $\alpha < 0 < \beta$

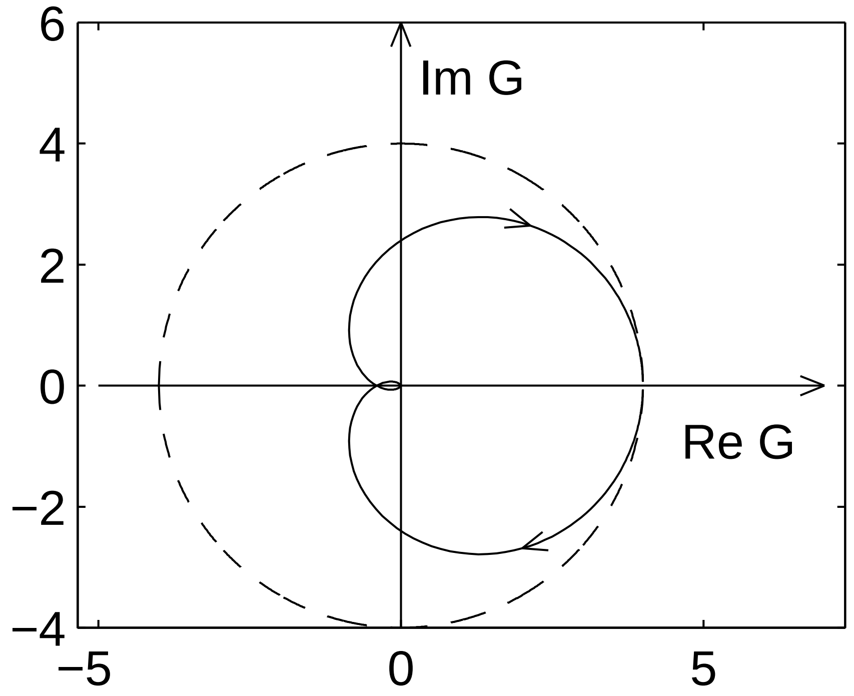
$$\mathrm{Re}\left[\frac{1+\beta G(j\omega)}{1+\alpha G(j\omega)}\right] > 0 \quad \Leftrightarrow \quad \mathrm{Re}\left[\frac{\frac{1}{\beta}+G(j\omega)}{\frac{1}{\alpha}+G(j\omega)}\right] < 0$$

The Nyquist plot of $G(j\omega)$ must lie inside the disk $D(\alpha, \beta)$.
The Nyquist plot cannot encircle the point $-(1/\alpha) + j0$.
From the Nyquist criterion, $G(s)$ must be Hurwitz

The system is absolutely stable if $G(s)$ is Hurwitz and the Nyquist plot of $G(j\omega)$ lies in the interior of the disk $D(\alpha, \beta)$

Example

$$G(s) = \frac{4}{(s+1)(\frac{1}{2}s+1)(\frac{1}{3}s+1)}$$



Apply Case 3 with center $(\mathbf{0}, \mathbf{0})$ and radius $= \mathbf{4}$

Sector is $(-\mathbf{0.25}, \mathbf{0.25})$

Apply Case 3 with center $(\mathbf{1.5}, \mathbf{0})$ and radius $= \mathbf{2.834}$

Sector is $[-\mathbf{0.227}, \mathbf{0.714}]$

Apply Case 2

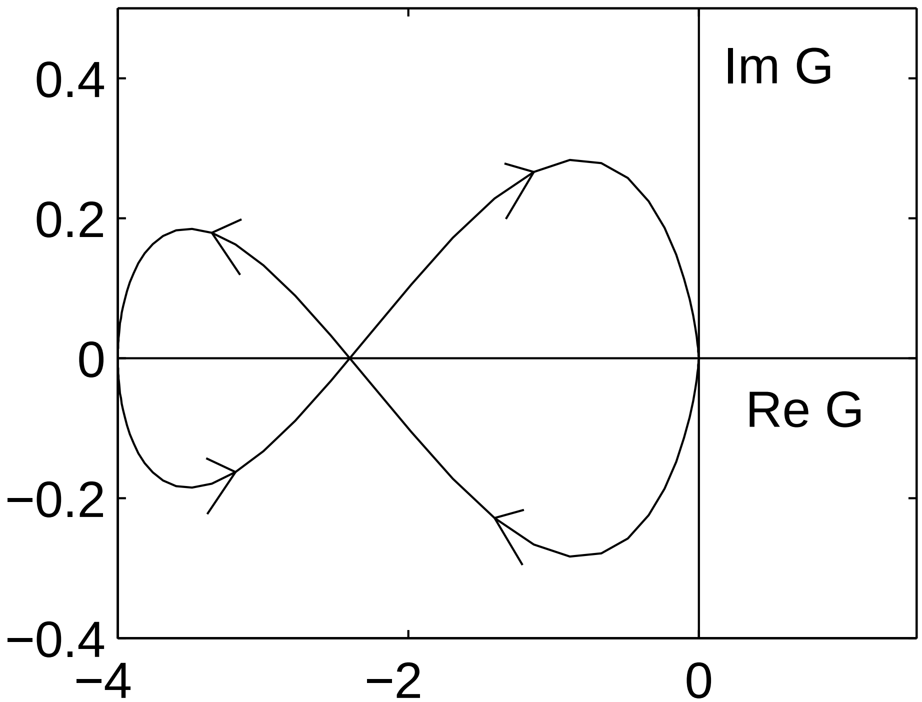
The Nyquist plot is to the right of $\mathbf{Re}[\boldsymbol{s}] = -\mathbf{0.857}$

Sector is $[\mathbf{0}, \mathbf{1.166}]$

$[\mathbf{0}, \mathbf{1.166}]$ includes the saturation nonlinearity

Example

$$G(s) = \frac{4}{(s-1)(\frac{1}{2}s+1)(\frac{1}{3}s+1)}$$

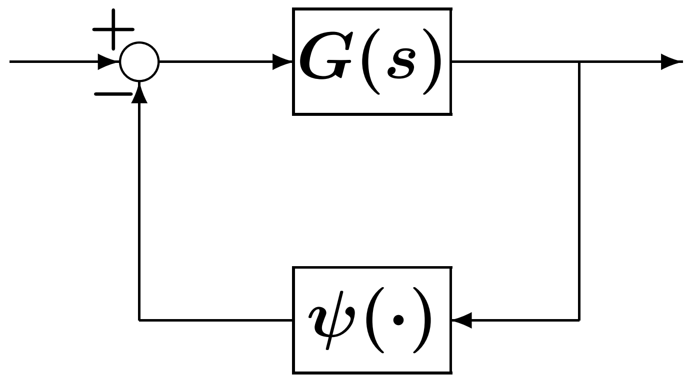


G is not Hurwitz

Apply Case 1

Center $= (-3.2, 0)$, Radius $= 0.168 \Rightarrow\ [0.2969, 0.3298]$

Popov Criterion

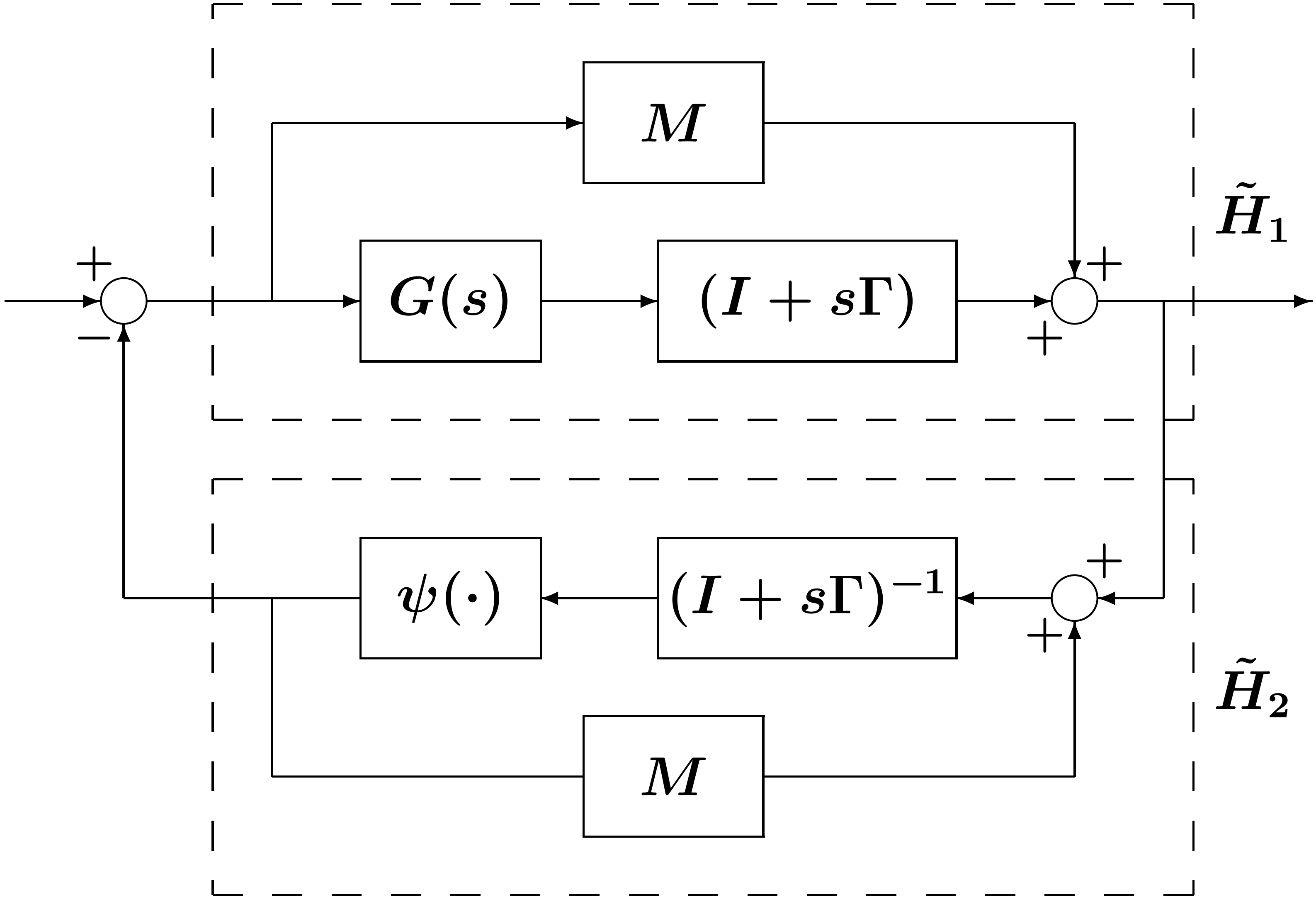


$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx \\ u_i &= -\psi_i(y_i), \;\; 1 \le i \le p\end{aligned}$$

$$\psi_i \in [0, k_i], \quad 1 \le i \le p, \quad (0 < k_i \le \infty)$$

$$G(s) = C(sI - A)^{-1}B$$

$$\Gamma = \mathrm{diag}(\gamma_1, \ldots, \gamma_p), \quad M = \mathrm{diag}(1/k_1, \cdots, 1/k_p)$$



Show that $\tilde{H}_1$ and $\tilde{H}_2$ are passive

$$
\begin{aligned}
M + (I + s\Gamma)G(s) &\\
&= M + (I + s\Gamma)C(sI - A)^{-1}B \\
&= M + C(sI - A)^{-1}B + \Gamma C s(sI - A)^{-1}B \\
&= M + C(sI - A)^{-1}B + \Gamma C(sI - A + A)(sI - A)^{-1}B \\
&= (C + \Gamma CA)(sI - A)^{-1}B + M + \Gamma CB
\end{aligned}
$$

If $M + (I + s\Gamma)G(s)$ is SPR, then $\tilde{H}_1$ is strictly passive with the storage function $V_1 = \frac{1}{2}x^T P x$, where P is given by the KYP equations

$$
\begin{aligned}
PA + A^T P &= -L^T L - \varepsilon P \\
PB &= (C + \Gamma CA)^T - L^T W \\
W^T W &= 2M + \Gamma CB + B^T C^T \Gamma
\end{aligned}
$$

$\tilde{H}_2$ consists of p decoupled components:

$$\gamma_i \dot{z}_i = -z_i + \frac{1}{k_i}\psi_i(z_i) + \tilde{e}_{2i}, \quad \tilde{y}_{2i} = \psi_i(z_i)$$

$$V_{2i} = \gamma_i \int_0^{z_i} \psi_i(\sigma)\ d\sigma$$

$$\begin{aligned} \dot{V}_{2i} &= \gamma_i \psi_i(z_i) \dot{z}_i = \psi_i(z_i) \left[-z_i + \tfrac{1}{k_i}\psi_i(z_i) + \tilde{e}_{2i} \right] \\ &= y_{2i} e_{2i} + \tfrac{1}{k_i}\psi_i(z_i) \left[\psi_i(z_i) - k_i z_i \right] \end{aligned}$$

$$\psi_i \in [0, k_i] \ \Rightarrow \ \psi_i(\psi_i - k_i z_i) \le 0 \ \Rightarrow \ \dot{V}_{2i} \le y_{2i} e_{2i}$$

$\tilde{H}_2$ is passive with the storage function
$V_2 = \sum_{i=1}^{p} \gamma_i \int_0^{z_i} \psi_i(\sigma)\ d\sigma$

Use $$V = \frac{1}{2}x^T P x + \sum_{i=1}^{p} \gamma_i \int_0^{y_i} \psi_i(\sigma)\, d\sigma$$

as a Lyapunov function candidate for the original feedback connection

$$\dot{x} = Ax + Bu, \quad y = Cx, \quad u = -\psi(y)$$

$$\begin{aligned}
\dot{V} &= \tfrac{1}{2}x^T P\dot{x} + \tfrac{1}{2}\dot{x}^T P x + \psi^T(y)\Gamma\dot{y} \\
&= \tfrac{1}{2}x^T(PA + A^T P)x + x^T PBu \\
&\quad + \psi^T(y)\Gamma C(Ax + Bu) \\
&= -\tfrac{1}{2}x^T L^T L x - \tfrac{1}{2}\varepsilon x^T P x \\
&\quad + x^T(C^T + A^T C^T \Gamma - L^T W)u \\
&\quad + \psi^T(y)\Gamma C A x + \psi^T(y)\Gamma C B u
\end{aligned}$$

$$
\begin{aligned}
\dot{V} &= -\tfrac{1}{2}\varepsilon x^T P x - \tfrac{1}{2}(Lx + Wu)^T(Lx + Wu) \\
&\quad - \psi(y)^T[y - M\psi(y)] \\
&\leq -\tfrac{1}{2}\varepsilon x^T P x - \psi(y)^T[y - M\psi(y)]
\end{aligned}
$$

$$
\psi_i \in [0, k_i] \;\Rightarrow\; \psi(y)^T[y - M\psi(y)] \geq 0 \;\Rightarrow\; \dot{V} \leq -\tfrac{1}{2}\varepsilon x^T P x
$$

The origin is globally asymptotically stable

Popov Criterion: The system is absolutely stable if, for $1 \leq i \leq p$, $\psi_i \in [0, k_i]$ and there exists a constant $\gamma_i \geq 0$, with $(1 + \lambda_k \gamma_i) \neq 0$ for every eigenvalue λ_k of A, such that $M + (I + s\Gamma)G(s)$ is strictly positive real

Scalar case

$$\frac{1}{k} + (1 + s\gamma)G(s)$$

is SPR if $G(s)$ is Hurwitz and

$$\frac{1}{k} + \mathrm{Re}[G(j\omega)] - \gamma\omega \mathrm{Im}[G(j\omega)] > 0, \quad \forall\, \omega \in [0, \infty)$$

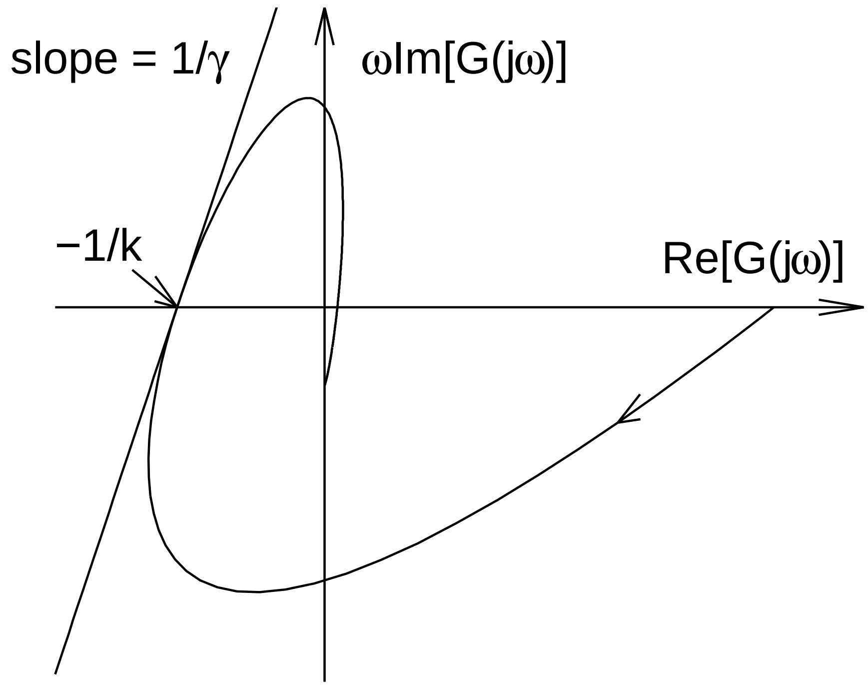
If

$$\lim_{\omega \to \infty} \left\{ \frac{1}{k} + \mathrm{Re}[G(j\omega)] - \gamma\omega \mathrm{Im}[G(j\omega)] \right\} = 0$$

we also need

$$\lim_{\omega \to \infty} \omega^2 \left\{ \frac{1}{k} + \mathrm{Re}[G(j\omega)] - \gamma\omega \mathrm{Im}[G(j\omega)] \right\} > 0$$

$$\frac{1}{k} + \mathrm{Re}[G(j\omega)] - \gamma\omega\mathrm{Im}[G(j\omega)] > 0, \quad \forall\, \omega \in [0, \infty)$$



Popov Plot

Example

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_2 - h(y), \quad y = x_1$$

$$\dot{x}_2 = -\alpha x_1 - x_2 - h(y) + \alpha x_1, \quad \alpha > 0$$

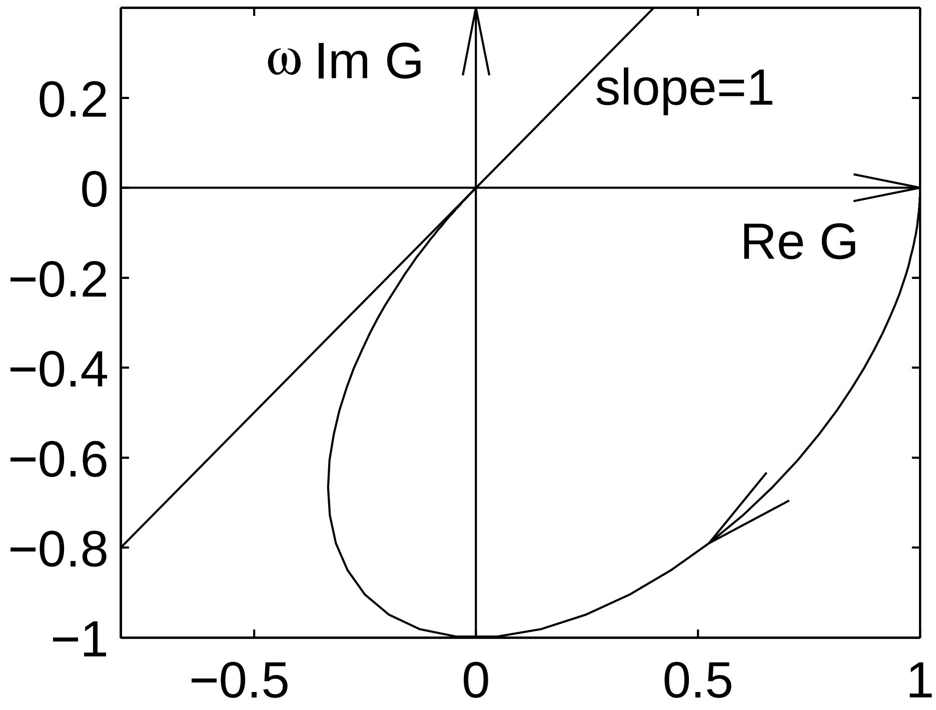
$$G(s) = \frac{1}{s^2 + s + \alpha}, \quad \psi(y) = h(y) - \alpha y$$

$$h \in [\alpha, \beta] \ \Rightarrow \ \psi \in [0, k] \qquad (k = \beta - \alpha > 0)$$

$$\gamma > 1 \ \Rightarrow \ \frac{\alpha - \omega^2 + \gamma\omega^2}{(\alpha - \omega^2)^2 + \omega^2} > 0, \ \forall\, \omega \in [0, \infty)$$

and $$\lim_{\omega \to \infty} \frac{\omega^2(\alpha - \omega^2 + \gamma\omega^2)}{(\alpha - \omega^2)^2 + \omega^2} = \gamma - 1 > 0$$

The system is absolutely stable for $\psi \in [0, \infty]$ ($h \in [\alpha, \infty]$)



Compare with the circle criterion ($\gamma = 0$)

$$\frac{1}{k} + \frac{\alpha - \omega^2}{(\alpha - \omega^2)^2 + \omega^2} > 0, \ \forall \omega \in [0, \infty], \quad \text{for } k < 1 + 2\sqrt{\alpha}$$