Nonlinear Systems and Control
Lecture # 29

Stabilization

Passivity-Based Control

$$\dot{x} = f(x,u), \qquad y = h(x)$$

$$f(0,0) = 0$$

$$u^T y \geq \dot{V} = \frac{\partial V}{\partial x} f(x,u)$$

Theorem 14.4: If the system is

(1) passive with a radially unbounded positive definite storage function and

(2) zero-state observable,

then the origin can be globally stabilized by

$$u = -\phi(y), \qquad \phi(0) = 0, \;\; y^T \phi(y) > 0 \; \forall \; y \neq 0$$

Proof:

$$\dot{V} = \frac{\partial V}{\partial x} f(x, -\phi(y)) \leq -y^T \phi(y) \leq 0$$

$$\dot{V}(x(t)) \equiv 0 \;\Rightarrow\; y(t) \equiv 0 \;\Rightarrow\; u(t) \equiv 0 \;\Rightarrow\; x(t) \equiv 0$$

Apply the invariance principle

A given system may be made passive by

(1) Choice of output,

(2) Feedback,

or both

Choice of Output

$$\dot{x} = f(x) + G(x)u, \qquad \frac{\partial V}{\partial x} f(x) \leq 0, \quad \forall\ x$$

No output is defined. Choose the output as

$$y = h(x) \stackrel{\text{def}}{=} \left[\frac{\partial V}{\partial x} G(x)\right]^T$$

$$\dot{V} = \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} G(x) u \leq y^T u$$

Check zero-state observability

Example

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -x_1^3 + u$$

$$V(x) = \tfrac{1}{4}x_1^4 + \tfrac{1}{2}x_2^2$$

With $u = 0 \qquad \dot{V} = x_1^3 x_2 - x_2 x_1^3 = 0$

Take $y = \dfrac{\partial V}{\partial x} G = \dfrac{\partial V}{\partial x_2} = x_2$

Is it zero-state observable?

with $u = 0, \qquad y(t) \equiv 0 \;\Rightarrow\; x(t) \equiv 0$

$u = -kx_2$ or $u = -(2k/\pi)\tan^{-1}(x_2) \qquad (k > 0)$

Feedback Passivation

Definition: The system

$$\dot{x} = f(x) + G(x)u, \qquad y = h(x)$$

is equivalent to a passive system if there is

$$u = \alpha(x) + \beta(x)v$$

such that

$$\dot{x} = f(x) + G(x)\alpha(x) + G(x)\beta(x)v, \qquad y = h(x)$$

is passive

Theorem [31]: The system

$$\dot{x} = f(x) + G(x)u, \qquad y = h(x)$$

is locally equivalent to a passive system (with a positive definite storage function) if it has relative degree one at $x = 0$ and the zero dynamics have a stable equilibrium point at the origin with a positive definite Lyapunov function

Example: m-link Robot Manipulator

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + D\dot{q} + g(q) = u$$

$$M = M^T > 0, \ (\dot{M} - 2C)^T = -(\dot{M} - 2C), \ D = D^T \geq 0$$

Stabilize the system at $q = q_r$

$$e = q - q_r, \qquad \dot{e} = \dot{q}$$

$$M(q)\ddot{e} + C(q,\dot{q})\dot{e} + D\dot{e} + g(q) = u$$

$(e = 0, \dot{e} = 0)$ is not an open-loop equilibrium point

$$u = g(q) - \phi_p(e) + v, \quad [\phi_p(0) = 0,\ e^T\phi_p(e) > 0\ \forall e \neq 0]$$

$$M(q)\ddot{e} + C(q,\dot{q})\dot{e} + D\dot{e} + \phi_p(e) = v$$

$$V = \tfrac{1}{2}\dot{e}^T M(q)\dot{e} + \int_0^e \phi_p^T(\sigma)\, d\sigma$$

$$\dot{V} = \tfrac{1}{2}\dot{e}^T(\dot{M} - 2C)\dot{e} - \dot{e}^T D\dot{e} - \dot{e}^T\phi_p(e) + \dot{e}^T v + \phi_p^T(e)\dot{e} \le \dot{e}^T v$$

$$y = \dot{e}$$

Is it zero-state observable? Set $v = 0$

$$\dot{e}(t) \equiv 0 \;\Rightarrow\; \ddot{e}(t) \equiv 0 \;\Rightarrow\; \phi_p(e(t)) \equiv 0 \;\Rightarrow\; e(t) \equiv 0$$

$$v = -\phi_d(\dot{e}), \qquad [\phi_d(0) = 0,\; \dot{e}^T \phi_d(\dot{e}) > 0 \; \forall \dot{e} \neq 0]$$

$$u = g(q) - \phi_p(e) - \phi_d(\dot{e})$$

Special case:

$$u = g(q) - K_p e - K_d \dot{e}, \qquad K_p = K_p^T > 0,\; K_d = K_d^T > 0$$

How does passivity-based control compare with feedback linearization?

Example 13.20

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -h(x_1) + u$$

$$h(0) = 0, \quad x_1 h(x_1) > 0, \ \forall \ x_1 \neq 0$$

Feedback linearization:

$$u = h(x_1) - (k_1 x_1 + k_2 x_2)$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix} x$$

Passivity-based control:

$$V = \int_0^{x_1} h(z)\, dz + \frac{1}{2}x_2^2$$

$$\dot{V} = x_2 h(x_1) - x_2 h(x_1) + x_2 u = x_2 u$$

Take $y = x_2$

With $u = 0$, $\quad y(t) \equiv 0 \;\Rightarrow\; h(x_1(t)) \equiv 0 \;\Rightarrow\; x_1(t) \equiv 0$

$$u = -\sigma(x_2), \qquad [\sigma(0) = 0,\; y\sigma(y) > 0 \;\forall\; y \neq 0]$$

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -h(x_1) - \sigma(x_2)$$

Linearization:

$$\begin{bmatrix} 0 & 1 \\ -h'(0) & -k \end{bmatrix}, \quad k = \sigma'(0)$$

$$s^2 + ks + h'(0) = 0$$

Sketch the root locus as k varies from zero to infinity

One of the two roots cannot be moved to the left of $\mathrm{Re}[s] = -\sqrt{h'(0)}$

Cascade Connection:

$$\dot{z} = f_a(z) + F(z,y)y, \quad \dot{x} = f(x) + G(x)u, \quad y = h(x)$$

$$f_a(0) = 0, \quad f(0) = 0, \quad h(0) = 0$$

$$\frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} G(x)u \le y^T u$$

$$\frac{\partial W}{\partial z} f_a(z) \le 0$$

$$U(z,x) = W(z) + V(x)$$

$$\dot{U} \le \frac{\partial W}{\partial z} F(z,y)y + y^T u \; = \; y^T \left[u + \left(\frac{\partial W}{\partial z} F(z,y) \right)^T \right]$$

$$u = -\left(\frac{\partial W}{\partial z} F(z,y)\right)^T + v \quad \Rightarrow \quad \dot{U} \le y^T v$$

The system

$$\begin{aligned} \dot{z} &= f_a(z) + F(z,y)y \\ \dot{x} &= f(x) - G(x)\left(\frac{\partial W}{\partial z} F(z,y)\right)^T + G(x)v \\ y &= h(x) \end{aligned}$$

with input v and output y is passive with U as the storage function

Read Examples 14.17 and 14.18