Nonlinear Systems and Control
Lecture # 31

Stabilization

Output Feedback

In general, output feedback stabilization requires the use of observers. In this lecture we deal with three simple cases where an observer is not needed

- Minimum Phase Relative Degree One Systems
- Passive systems
- System with Passive maps from the input to the derivative of the output

Minimum Phase Relative Degree One Systems

$$\dot{x} = f(x) + g(x)u, \qquad y = h(x)$$

$$f(0) = 0, \quad h(0) = 0, \quad L_g h(x) \neq 0, \quad \forall\, x \in D$$

Normal Form:

$$\phi(0) = 0, \quad L_g \phi(x) = 0, \quad \begin{bmatrix} \eta \\ y \end{bmatrix} = \begin{bmatrix} \phi \\ h(x) \end{bmatrix}$$

$$\dot{\eta} = f_0(\eta, y), \quad \dot{y} = \gamma(x)[u - \alpha(x)], \quad \gamma(x) \neq 0$$

Assumptions:
The origin of $\dot{\eta} = f_0(\eta, 0)$ is exponentially stable

$$c_1\|\eta\|^2 \le V_1(\eta) \le c_2\|\eta\|^2$$

$$\frac{\partial V_1}{\partial \eta} f_0(\eta, 0) \le -c_3\|\eta\|^2$$

$$\left|\frac{\partial V_1}{\partial \eta}\right| \le c_4\|\eta\|$$

$$\|f_0(\eta, y) - f_0(\eta, 0)\| \le L_1|y|$$

$$|\alpha(x)\gamma(x)| \le L_2\|\eta\| + L_3|y|$$

$$\gamma(x) \ge \gamma_0 > 0$$

High-Gain Feedback:

$$u = -ky, \quad k > 0$$

$$\dot{\eta} = f_0(\eta, y), \quad \dot{y} = \gamma(x)[-ky - \alpha(x)]$$

$$V(\eta, y) = V_1(\eta) + \tfrac{1}{2}y^2$$

$$\dot{V} = \frac{\partial V_1}{\partial \eta} f_0(\eta, y) - k\gamma(x)y^2 - \alpha(x)\gamma(x)y$$

$$\begin{aligned} \dot{V} \ &= \ \frac{\partial V_1}{\partial \eta} f_0(\eta, 0) + \frac{\partial V_1}{\partial \eta}\left[f_0(\eta, y) - f_0(\eta, 0)\right] \\ &\quad - k\gamma(x)y^2 - \alpha(x)\gamma(x)y \end{aligned}$$

$$\dot{V} \leq -c_3\|\eta\|^2 + c_4L_1\|\eta\| \, |y| - k\gamma_0 y^2 + L_2\|\eta\| \, |y| + L_3 y^2$$

$$\dot{V} \leq -c_3\|\eta\|^2 + c_4 L_1 \|\eta\| \, |y| - k\gamma_0 y^2 + L_2 \|\eta\| \, |y| + L_3 y^2$$

$$\dot{V} \leq - \begin{bmatrix} \|\eta\| \\ |y| \end{bmatrix}^T \underbrace{\begin{bmatrix} c_3 & -\frac{1}{2}(L_2 + c_4 L_1) \\ -\frac{1}{2}(L_2 + c_4 L_1) & (k\gamma_0 - L_3) \end{bmatrix}}_{Q} \begin{bmatrix} \|\eta\| \\ |y| \end{bmatrix}$$

$$\det(Q) = c_3(k\gamma_0 - L_3) - \tfrac{1}{4}(L_2 + c_4 L_1)^2$$

$$\det(Q) > 0 \quad \text{for} \quad k > \frac{1}{c_3\gamma_0}\left[c_3 L_3 + \tfrac{1}{4}(L_2 + c_4 L_1)^2\right]$$

The origin of the closed-loop system is exponentially stable

If the assumptions hold globally, it is globally exp. stable

Passive Systems

Suppose the system

$$\dot{x} = f(x, u), \quad y = h(x, u)$$

is passive (with a positive definite storage function) and zero-state observable

Then, it can be stabilized by

$$u = -\psi(y), \quad \psi(0) = 0, \; y^T \psi(y) > 0, \; \forall \; y \neq 0$$

$$\dot{V} \leq u^T y = -y^T \psi(y) \leq 0$$

$$\dot{V} = 0 \;\Rightarrow\; y = 0 \;\Rightarrow\; u = 0$$

$$y(t) \equiv 0 \;\;\Rightarrow\;\; x(t) \equiv 0$$

Systems with Passive Maps from u to $\dot{y}$

$$\dot{x} = f(x,u), \qquad y = h(x)$$

$$f(0,0) = 0, \quad h(0) = 0, \quad h \text{ is cont. diff.}$$

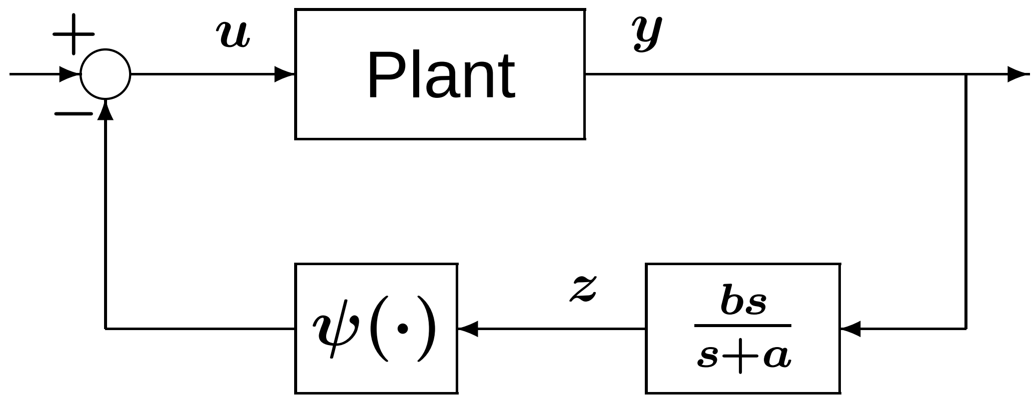
Suppose the system

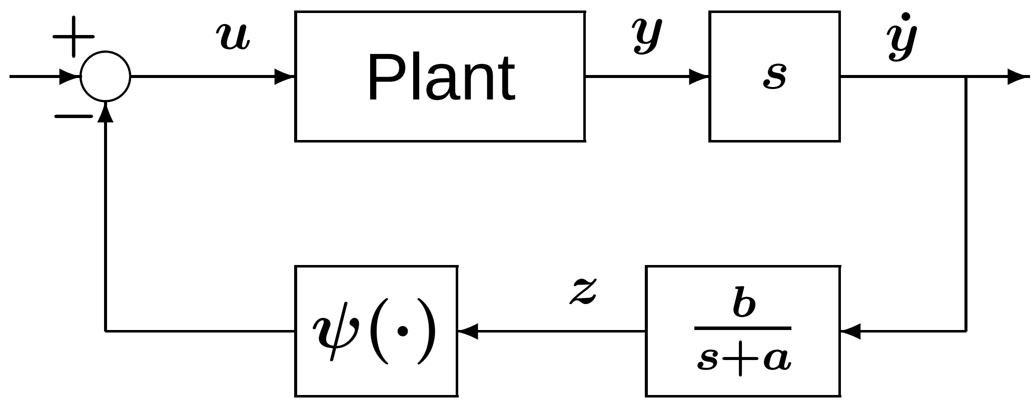
$$\dot{x} = f(x,u), \quad \dot{y} = \frac{\partial h}{\partial x} f(x,u) \stackrel{\text{def}}{=} \tilde{h}(x,u)$$

is passive (with a positive definite storage function) and zero-state observable

$$\dot{V} \le u^T \dot{y}$$

$$\text{With } u = 0, \quad \dot{y}(t) \equiv 0 \;\; \Rightarrow \;\; x(t) \equiv 0$$


+
−
u
Plant
y
z
ψ(·)
bs/(s+a)


+
−
u
Plant
y
s
ẏ
z
ψ(·)
b/(s+a)

For $1 \leq i \leq m$

$$z_i \text{ is the output of } \frac{b_i s}{s + a_i} \text{ driven by } \; y_i$$

$$u_i = -\psi_i(z_i)$$

$$a_i, b_i > 0, \quad \psi_i(0) = 0, \quad z_i \psi_i(z_i) > 0 \;\; \forall \; z_i \neq 0$$

$$\dot{z}_i = -a_i z_i + b_i \dot{y}_i$$

Use

$$V_c(x, z) = V(x) + \sum_{i=1}^{m} \frac{1}{b_i} \int_0^{z_i} \psi_i(\sigma) \; d\sigma$$

to prove asymptotic stability of the origin of the closed-loop system

Example: Stabilize the pendulum

$$m\ell\ddot{\theta} + mg\sin\theta = u$$

at $\theta = \delta_1$ using feedback from θ

$$x_1 = \theta - \delta_1, \qquad x_2 = \dot{\theta}$$

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = \frac{1}{m\ell}[-mg\sin\theta + u]$$

$$u = mg\sin\theta - k_p x_1 + v, \qquad k_p > 0$$

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\frac{k_p}{m\ell}x_1 + \frac{1}{m\ell}v$$

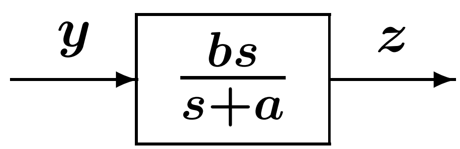
$$y = x_1, \qquad \dot{y} = x_2$$

$$V = \frac{1}{2}k_p x_1^2 + \frac{1}{2}m\ell x_2^2$$

$$\dot{V} = k_p x_1 x_2 + m\ell x_2 \left[-\frac{k_p}{m\ell} x_1 + \frac{1}{m\ell} v \right]$$

$$\dot{V} = x_2 v = \dot{y} v$$

With $v = 0$, $\quad x_2(t) \equiv 0 \quad \Rightarrow \quad x_1(t) \equiv 0$



$$\dot{\xi} = -a\xi + y, \qquad z = b(-a\xi + y)$$

$$v = -k_d z, \qquad k_d > 0$$

$$u = mg \sin\theta - k_p(\theta - \delta_1) - k_d z$$