Nonlinear Systems and Control
Lecture # 35

Tracking

Feedback Linearization &
Sliding Mode Control

SISO relative-degree ρ system:

$$\dot{x} = f(x) + g(x)u, \qquad y = h(x)$$

$$f(0) = 0, \quad h(0) = 0$$

$$L_g L_f^{i-1} h(x) = 0, \;\; \text{for } 1 \le i \le \rho - 1, \quad L_g L_f^{\rho-1} h(x) \ne 0$$

Normal form:

$$\begin{aligned} \dot{\eta} &= f_0(\eta, \xi) \\ \dot{\xi}_i &= \xi_{i+1}, \quad 1 \le i \le \rho - 1 \\ \dot{\xi}_\rho &= L_f^\rho h(x) + L_g L_f^{\rho-1} h(x) u \\ y &= \xi_1 \end{aligned}$$

$$f_0(0,0) = 0$$

Reference signal $r(t)$

- $r(t)$ and its derivatives up to $r^{(\rho)}(t)$ are bounded for all $t \geq 0$ and the ρth derivative $r^{(\rho)}(t)$ is a piecewise continuous function of t;
- the signals r,...,$r^{(\rho)}$ are available on-line.

Goal: $\lim_{t\to\infty} [y(t) - r(t)] = 0$

$$\mathcal{R} = \begin{bmatrix} r \\ \vdots \\ r^{(\rho-1)} \end{bmatrix}, \quad e = \begin{bmatrix} \xi_1 - r \\ \vdots \\ \xi_\rho - r^{(\rho-1)} \end{bmatrix} = \xi - \mathcal{R}$$

$$
\begin{aligned}
\dot{\eta} &= f_0(\eta, e + \mathcal{R}) \\
\dot{e} &= A_c e + B_c \left[L_f^{\rho} h(x) + L_g L_f^{\rho-1} h(x) u - r^{(\rho)} \right]
\end{aligned}
$$

$$
A_c = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & 0 & 1 \\ 0 & \dots & \dots & 0 & 0 \end{bmatrix}, \; B_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}
$$

$$
u = \frac{1}{L_g L_f^{\rho-1} h(x)} \left[-L_f^{\rho} h(x) + r^{(\rho)} + v \right]
$$

$$
\dot{e} = A_c e + B_c v
$$

$$v = -Ke \quad \Rightarrow \quad \dot{e} = \underbrace{(A_c - B_c K)}_{Hurwitz} e$$

$$\lim_{t \to \infty} e(t) = 0 \quad \Rightarrow \quad \lim_{t \to \infty} [y(t) - r(t)] = 0$$

$e(t)$ is bounded $\quad \Rightarrow \quad \xi(t) = e(t) + \mathcal{R}(t)$ is bounded

What about $\eta(t)$?

$$\dot{\eta} = f_0(\eta, \xi)$$

Local Tracking (small $\|\eta(0)\|$, $\|e(0)\|$, $\|\mathcal{R}(t)\|$):

Minimum Phase $\Rightarrow$ The origin of $\dot{\eta} = f_0(\eta, 0)$ is asymptotically stable

$\Rightarrow$ η is bounded for sufficiently small $\|\eta(0)\|$, $\|e(0)\|$, and $\|\mathcal{R}(t)\|$

Global Tracking (large $\|\eta(0)\|$, $\|e(0)\|$, $\|\mathcal{R}(t)\|$):

What condition on $\dot{\eta} = f_0(\eta, \xi)$ is needed?

Example 13.21

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -a\sin x_1 - bx_2 + cu, \quad y = x_1$$

$$e_1 = x_1 - r, \quad e_2 = x_2 - \dot{r}$$

$$\dot{e}_1 = e_2, \quad \dot{e}_2 = -a\sin x_1 - bx_2 + cu - \ddot{r}$$

$$u = \frac{1}{c}[a\sin x_1 + bx_2 + \ddot{r} - k_1e_1 - k_2e_2]$$

$$\dot{e}_1 = e_2, \quad \dot{e}_2 = -k_1e_1 - k_2e_2$$

See simulation in the textbook

Sliding Mode Control

$$\dot{x} = f(x) + g(x)[u + \delta(t,x,u)], \quad y = h(x)$$

$$L_g h(x) = \cdots = L_g L_f^{\rho-2} h(x) = 0, \quad L_g L_f^{\rho-1} h(x) \geq a > 0$$

$$\begin{aligned}
\dot{\eta} &= f_0(\eta, \xi) \\
\dot{\xi}_1 &= \xi_2 \\
\vdots \quad & \quad \vdots \\
\dot{\xi}_{\rho-1} &= \xi_\rho \\
\dot{\xi}_\rho &= L_f^\rho h(x) + L_g L_f^{\rho-1} h(x)[u + \delta(t,x,u)] \\
y &= \xi_1
\end{aligned}$$

$$e = \xi - \mathcal{R}$$

$$
\begin{aligned}
\dot{\eta} &= f_0(\eta, \xi) \\
\dot{e}_1 &= e_2 \\
&\ \ \vdots \\
\dot{e}_{\rho-1} &= e_\rho \\
\dot{e}_\rho &= L_f^\rho h(x) + L_g L_f^{\rho-1} h(x)[u + \delta(t, x, u)] - r^{(\rho)}(t)
\end{aligned}
$$

Sliding surface:

$$
s = (k_1 e_1 + \cdots + k_{\rho-1} e_{\rho-1}) + e_\rho
$$

$$
s(t) \equiv 0 \;\; \Rightarrow \;\; e_\rho = -(k_1 e_1 + \cdots + k_{\rho-1} e_{\rho-1})
$$

$$
\begin{aligned}
\dot{\eta} &= f_0(\eta, \xi) \\
\dot{e}_1 &= e_2 \\
\vdots & \quad \vdots \\
\dot{e}_{\rho-1} &= -(k_1 e_1 + \cdots + k_{\rho-1} e_{\rho-1})
\end{aligned}
$$

Design k_1 to $k_{\rho-1}$ such that the matrix

$$
\begin{bmatrix}
 & 1 & & \\
 & & \ddots & \\
 & & & 1 \\
-k_1 & & & -k_{\rho-1}
\end{bmatrix}
\text{ is Hurwitz}
$$

Assumption: The system $f_0(\eta, \xi)$ is BIBS stable

$$s = (k_1 e_1 + \cdots + k_{\rho-1} e_{\rho-1}) + e_\rho = \sum_{i=1}^{\rho-1} k_i e_i + e_\rho$$

$$\dot{s} = \sum_{i=1}^{\rho-1} k_i e_{i+1} + L_f^\rho h(x) + L_g L_f^{\rho-1} h(x)[u + \delta(t,x,u)] - r^{(\rho)}(t)$$

$$u = -\frac{1}{L_g L_f^{\rho-1} h(x)} \left[\sum_{i=1}^{\rho-1} k_i e_{i+1} + L_f^\rho h(x) - r^{(\rho)}(t) \right] + v$$

$$\dot{s} = L_g L_f^{\rho-1} h(x) v + \Delta(t,x,v)$$

$$\left| \frac{\Delta(t,x,v)}{L_g L_f^{\rho-1} h(x)} \right| \le \varrho(x) + \kappa_0 |v|, \quad 0 \le \kappa_0 < 1$$

$$v = -\beta(x)\,\mathrm{sat}\left(\frac{s}{\varepsilon}\right), \qquad \varepsilon > 0$$

$$\beta(x) \geq \frac{\varrho(x)}{(1-\kappa_0)} + \beta_0, \qquad \beta_0 >$$

What properties can we prove for this control?