Nonlinear Systems and Control
Lecture # 5
Limit Cycles

Oscillation: A system oscillates when it has a nontrivial periodic solution

$$x(t+T) = x(t), \;\; \forall\, t \geq 0$$

Linear (Harmonic) Oscillator:

$$\dot{z} = \begin{bmatrix} 0 & -\beta \\ \beta & 0 \end{bmatrix} z$$

$$z_1(t) = r_0 \cos(\beta t + \theta_0), \qquad z_2(t) = r_0 \sin(\beta t + \theta_0)$$

$$r_0 = \sqrt{z_1^2(0) + z_2^2(0)}, \qquad \theta_0 = \tan^{-1}\left[\frac{z_2(0)}{z_1(0)}\right]$$

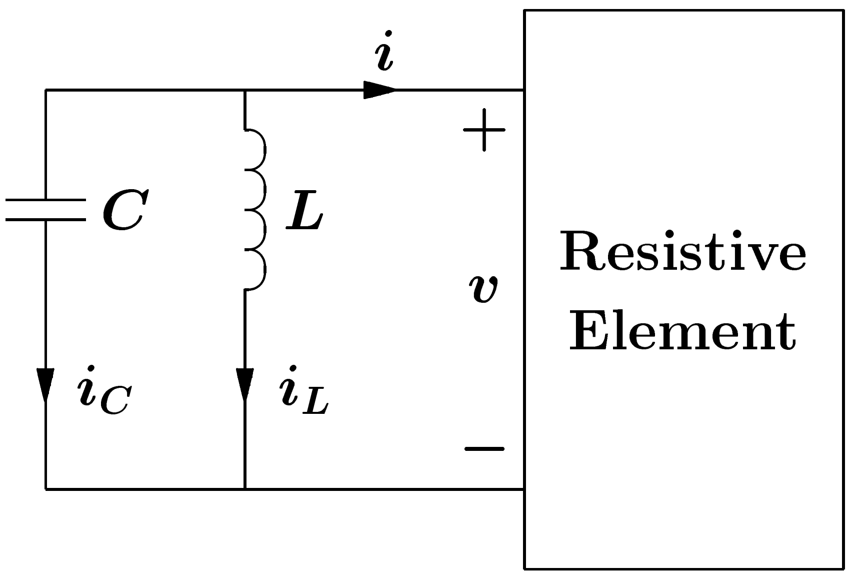
The linear oscillation is not practical because

- It is not structurally stable. Infinitesimally small perturbations may change the type of the equilibrium point to a stable focus (decaying oscillation) or unstable focus (growing oscillation)
- The amplitude of oscillation depends on the initial conditions

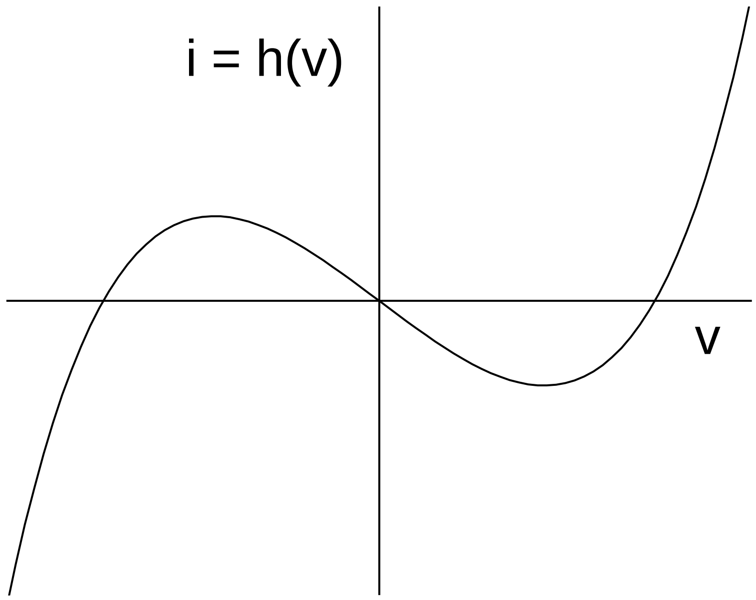
The same problems exist with oscillation of nonlinear systems due to a center equilibrium point (e.g., pendulum without friction)

Limit Cycles:

Example: Negative Resistance Oscillator



(a)



(b)

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 - \varepsilon h'(x_1) x_2
\end{aligned}
$$

There is a unique equilibrium point at the origin

$$
A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} 0 & 1 \\ -1 & -\varepsilon h'(0) \end{bmatrix}
$$

$$
\lambda^2 + \varepsilon h'(0)\lambda + 1 = 0
$$

$h'(0) < 0 \;\Rightarrow$ Unstable Focus or Unstable Node

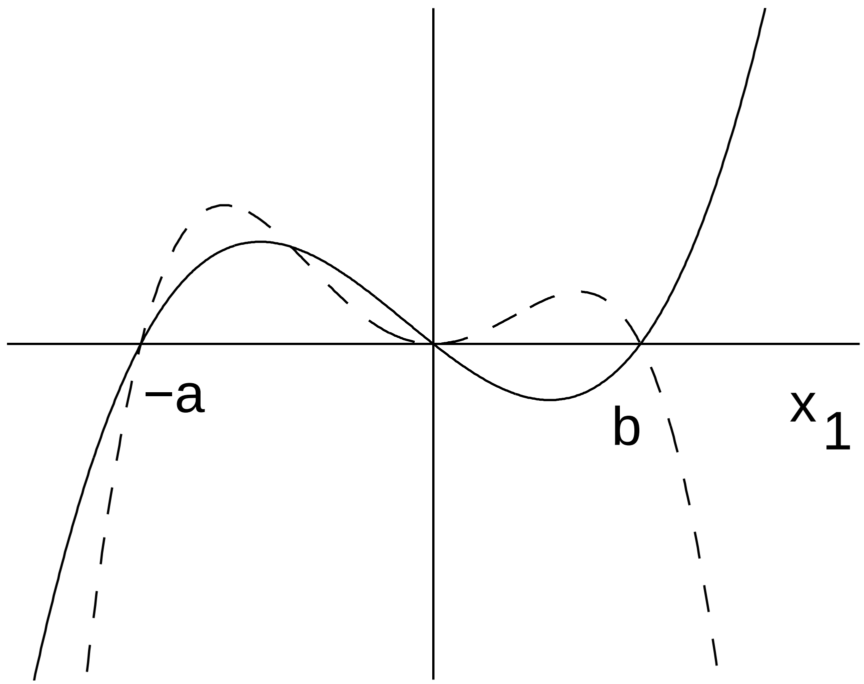
Energy Analysis:

$$E = \tfrac{1}{2}Cv_C^2 + \tfrac{1}{2}Li_L^2$$

$$v_C = x_1 \quad \text{and} \quad i_L = -h(x_1) - \frac{1}{\varepsilon}x_2$$

$$E = \tfrac{1}{2}C\{x_1^2 + [\varepsilon h(x_1) + x_2]^2\}$$

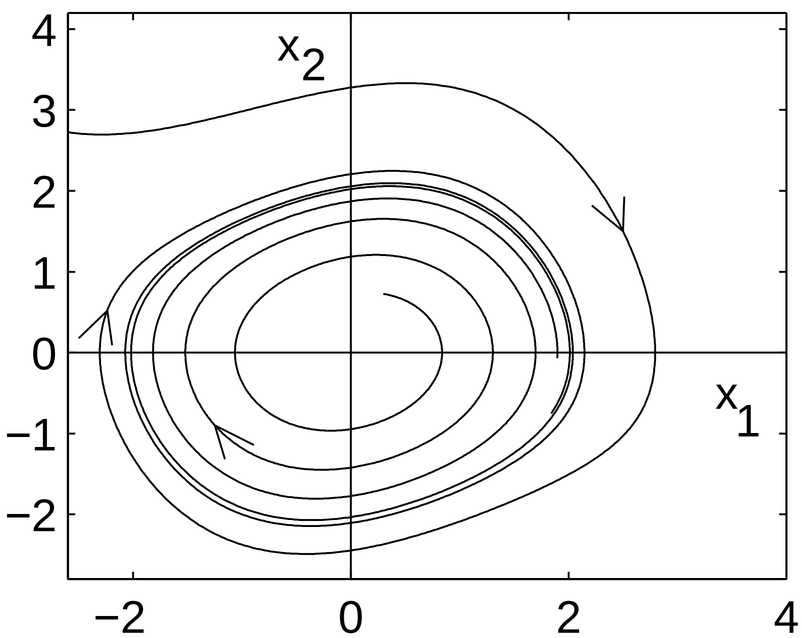
$$\begin{aligned}
\dot{E} &= C\{x_1\dot{x}_1 + [\varepsilon h(x_1) + x_2][\varepsilon h'(x_1)\dot{x}_1 + \dot{x}_2]\} \\
&= C\{x_1x_2 + [\varepsilon h(x_1) + x_2][\varepsilon h'(x_1)x_2 - x_1 - \varepsilon h'(x_1)x_2]\} \\
&= C[x_1x_2 - \varepsilon x_1 h(x_1) - x_1x_2] \\
&= -\varepsilon C x_1 h(x_1)
\end{aligned}$$



$$\dot{\boldsymbol{E}} = -\boldsymbol{\varepsilon C x_1 h(x_1)}$$

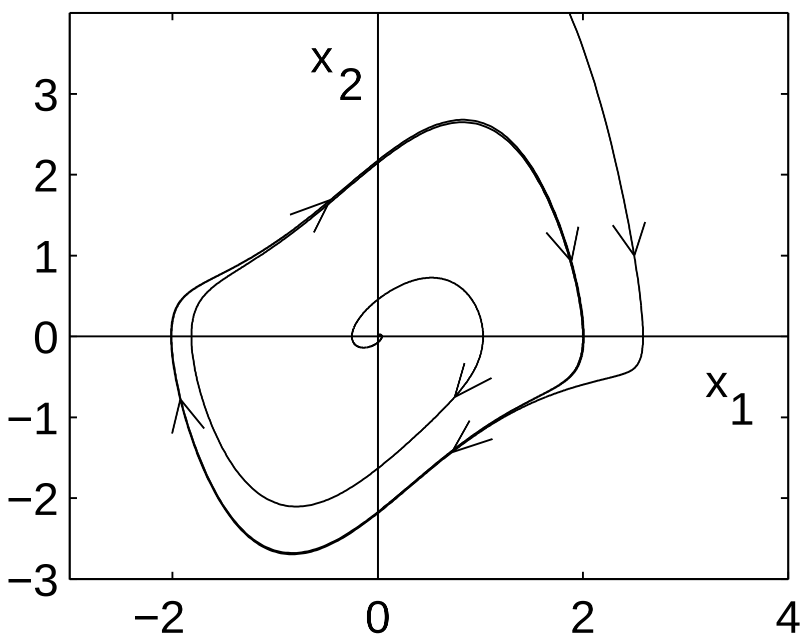
Example: Van der Pol Oscillator

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \varepsilon(1 - x_1^2)x_2 \end{aligned}$$



(a)

$\varepsilon = 0.2$

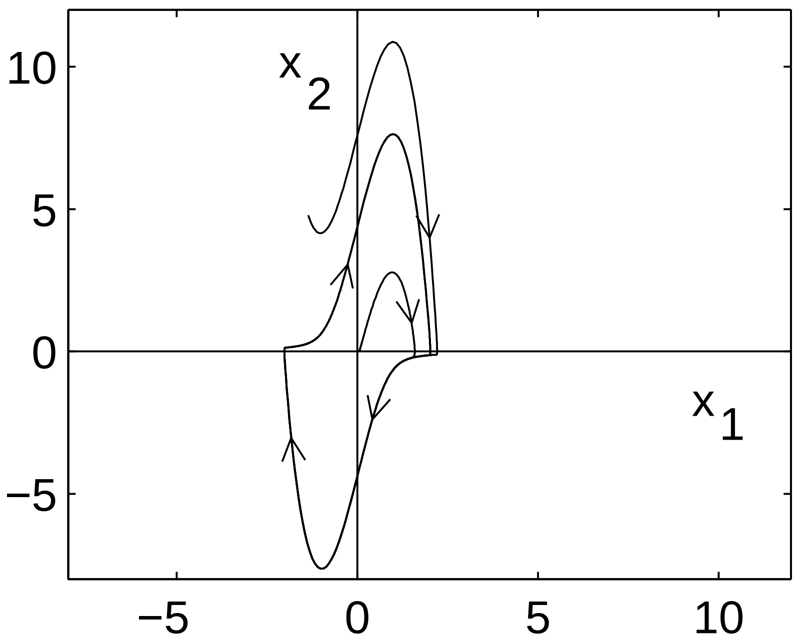


(b)

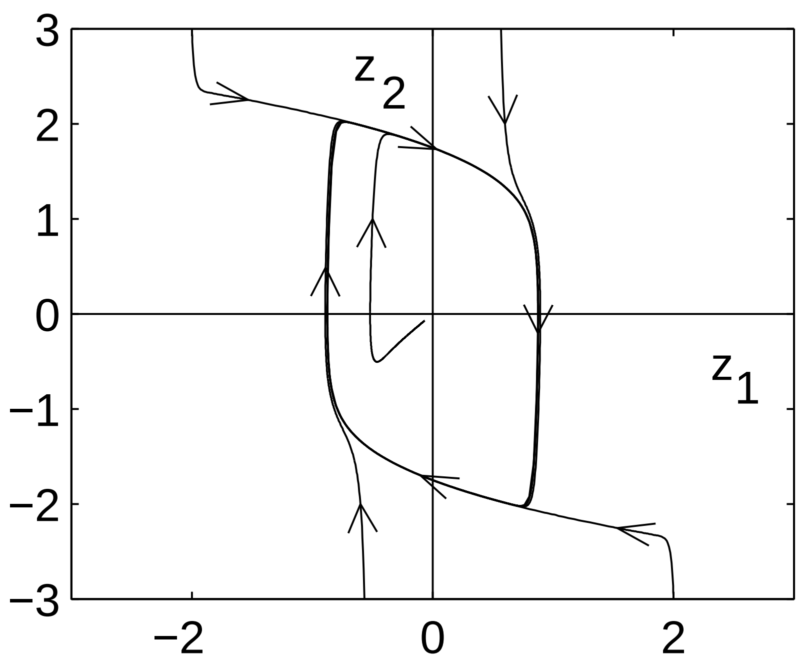
$\varepsilon = 1$

$$\dot{z}_1 = \frac{1}{\varepsilon} z_2$$

$$\dot{z}_2 = -\varepsilon\left(z_1 - z_2 + \tfrac{1}{3} z_2^3\right)$$

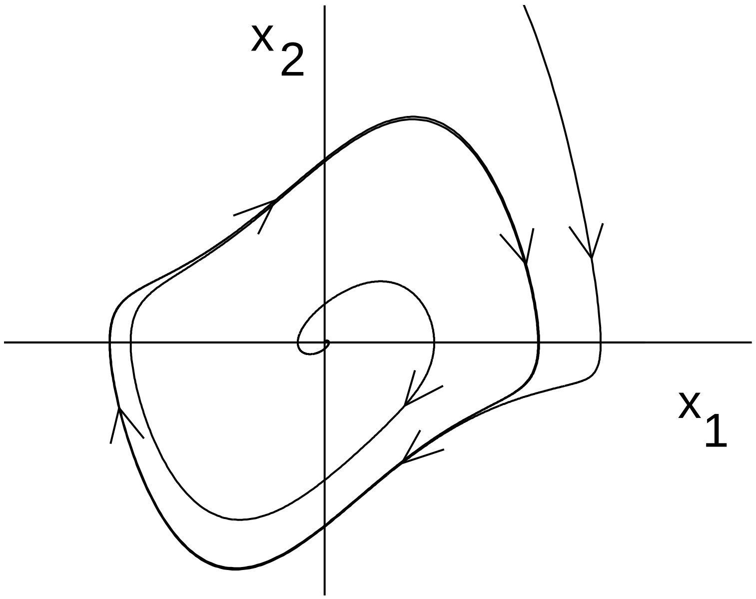


(a)

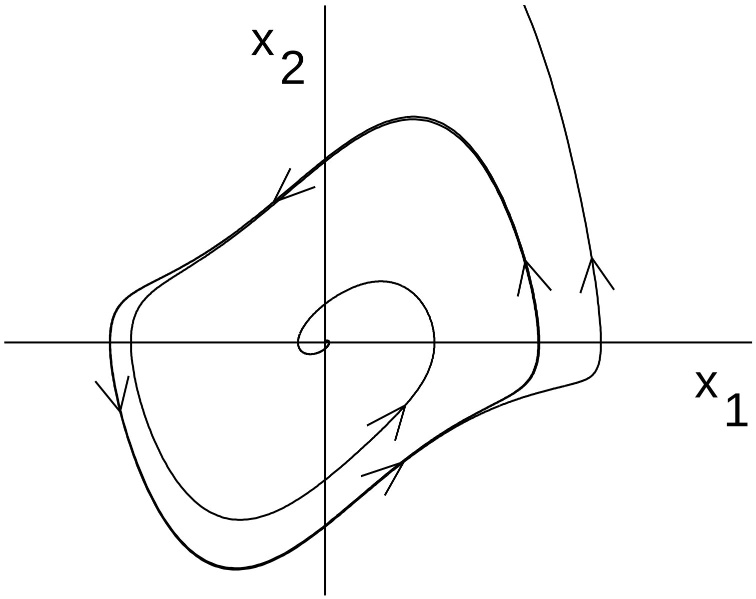


(b)

$\varepsilon = 5$



(a)

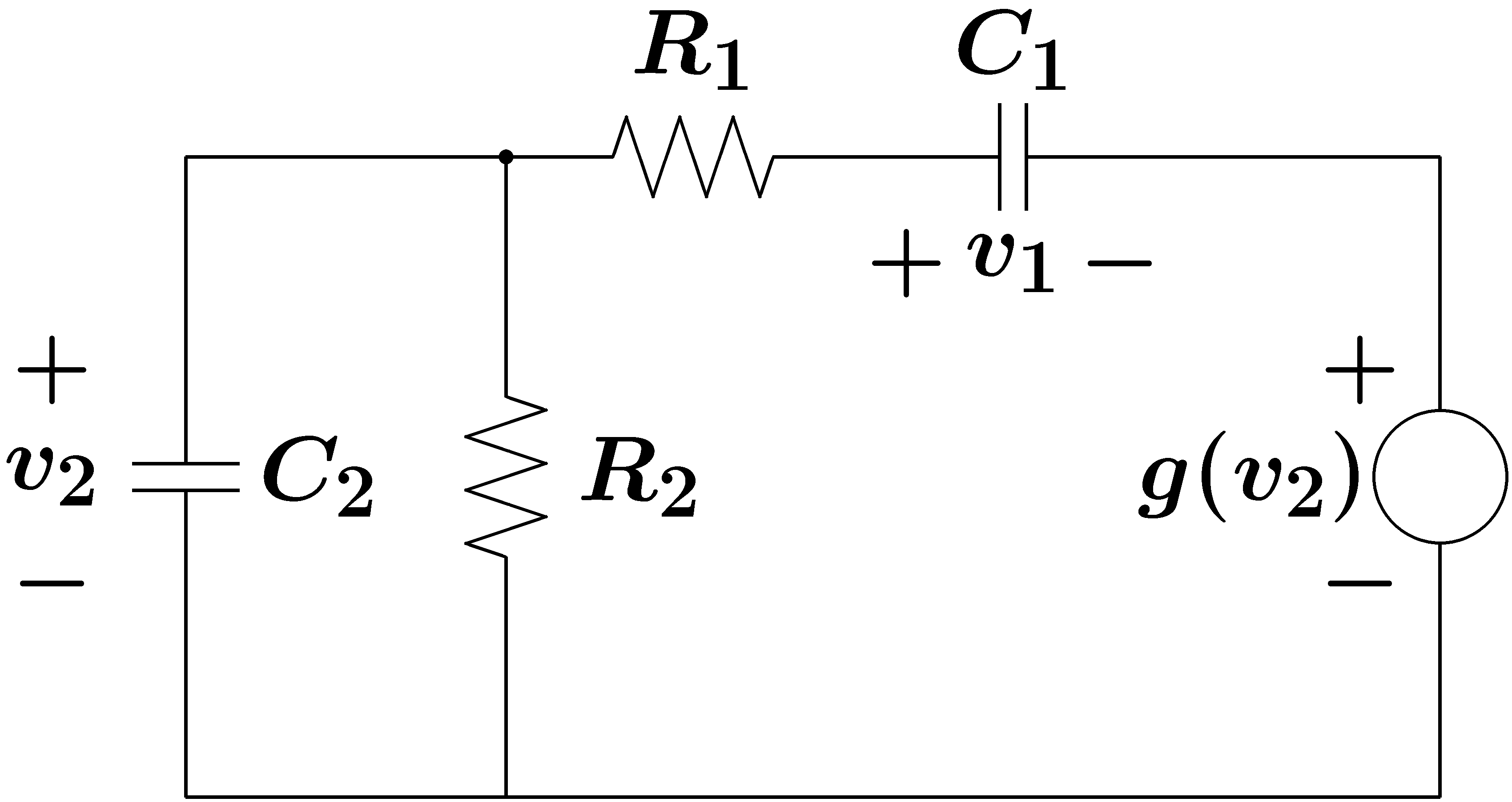


(b)

Stable Limit Cycle

Unstable Limit Cycle

Example: Wien-Bridge Oscillator



Equivalent Circuit

State variables $x_1 = v_1$ and $x_2 = v_2$

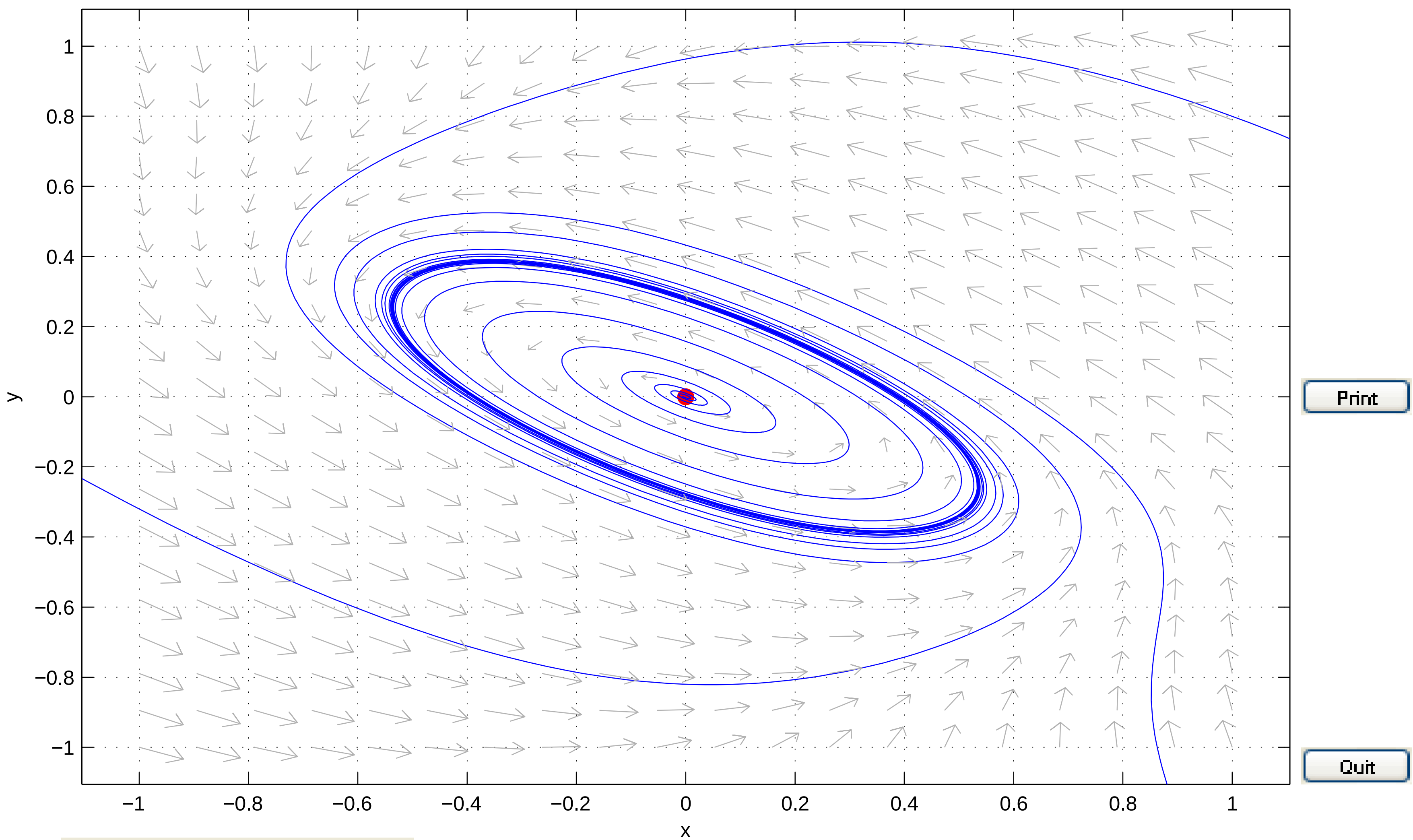
$$
\begin{aligned}
\dot{x}_1 &= \frac{1}{C_1 R_1}[-x_1 + x_2 - g(x_2)] \\
\dot{x}_2 &= -\frac{1}{C_2 R_1}[-x_1 + x_2 - g(x_2)] - \frac{1}{C_2 R_2} x_2
\end{aligned}
$$

There is a unique equilibrium point at $x = 0$

Numerical data: $C_1 = C_2 = R_1 = R_2 = 1$

$$g(v) = 3.234v - 2.195v^3 + 0.666v^5$$

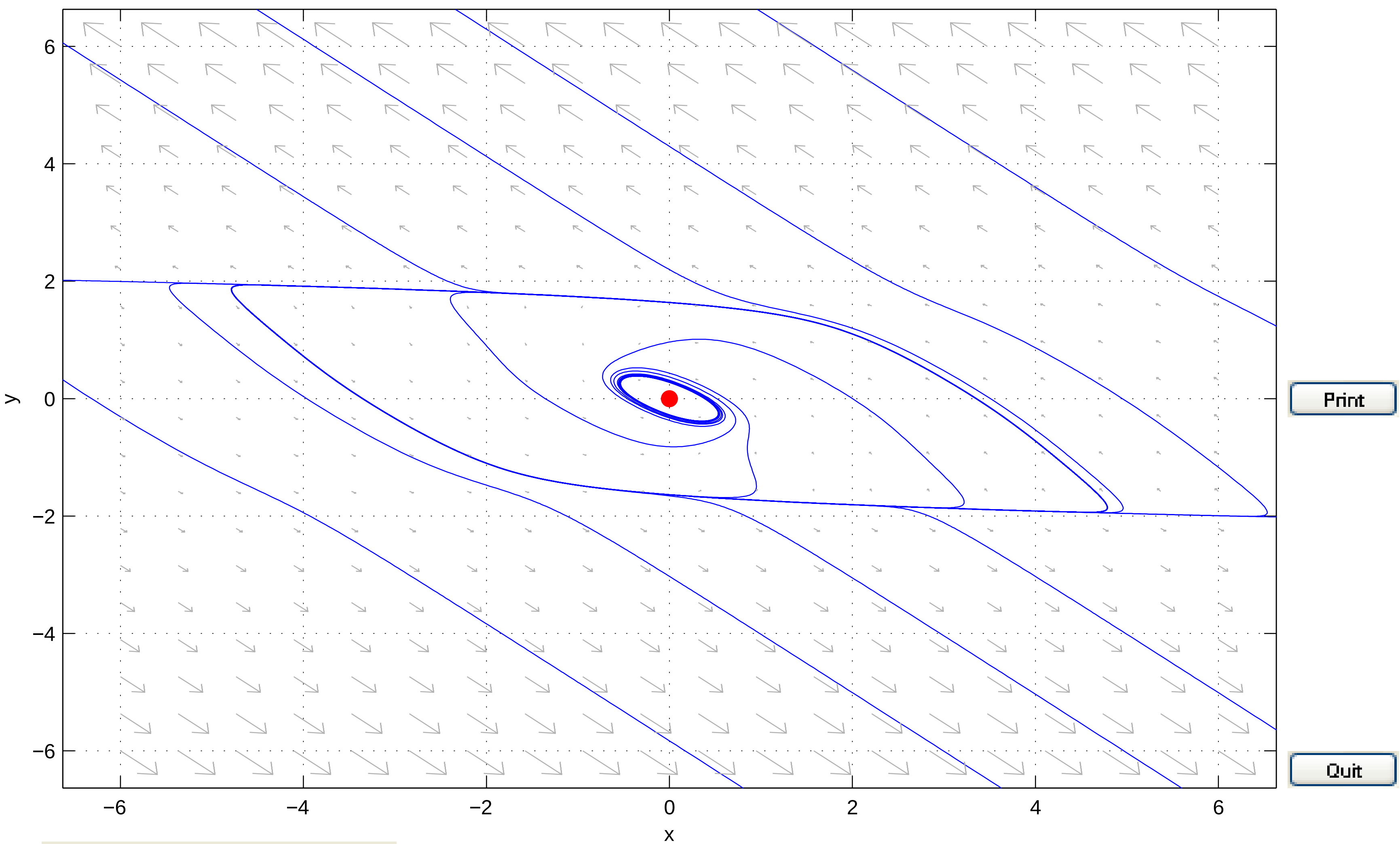
$$x' = -x + y - (3.234\,y - 2.195\,y^3 + 0.666\,y^5)$$
$$y' = -(-x + y - (3.234\,y - 2.195\,y^3 + 0.666\,y^5)) - y$$



Cursor position: (-0.663, -1.72)

Computing the field elements.
Ready.
The forward orbit from (0.13, -0.1) --> a nearly closed orbit.
The backward orbit from (0.13, -0.1) --> a possible eq. pt. near (0, 0).
Ready.

$x' = -x + y - (3.234\, y - 2.195\, y^3 + 0.666\, y^5)$
$y' = -(-x + y - (3.234\, y - 2.195\, y^3 + 0.666\, y^5)) - y$



Cursor position: (-5.72, -8.7)

Ready.
Computing the field elements.
Ready.
Select a graphics object with the mouse.
Ready.